Some questions on free-by-cyclic groups and tubular groups

Shengkui Ye

New York University Shanghai

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Outline

- What are free-by-cyclic groups and tubular groups?

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- Solutions

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- Questions
- Solutions
- Idea of proofs

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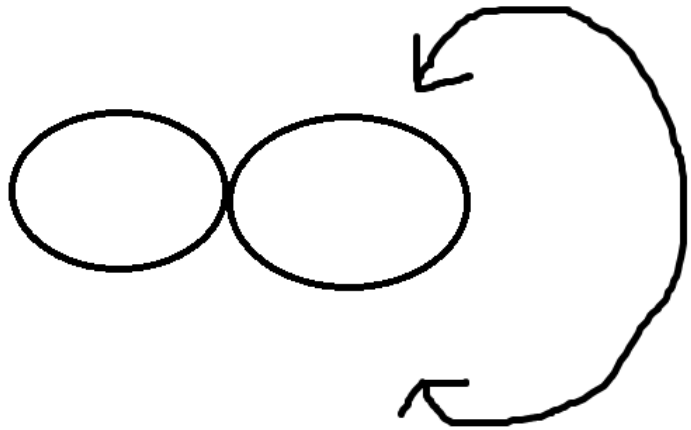

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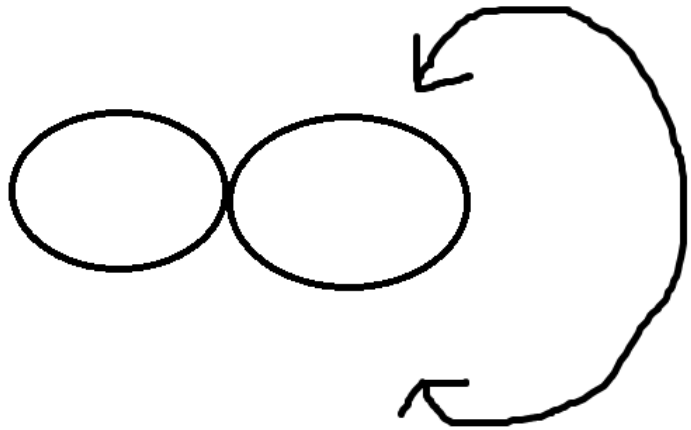

- Anolog of surface bundle over the circle, 3-manifold groups.

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- Problem list: http://aimpl.org/freebycyclic/

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- For example, $\mathbb{Z}^n, F_n$, virtually special groups have VRC.
- [Minasyan] Let G be the fundamental group of a finite graph of groups with free abelian vertex groups (of finite ranks) and cyclic edge groups. If G is cyclic subgroup separable, does it necessarily satisfy (VRC)?

Solutions

- [Hagen-Wise] Is the amalgamated product of two free-by-cyclic groups over a cyclic subgroup (virtually) free-by-cyclic?

Theorem 1

Let $G_1 = \langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = ba, tct^{-1} = ca^2 \rangle$ *be the Gersten group,* $G_2 = \langle x, s \mid sxs^{-1} = x \rangle \cong \mathbb{Z}^2$. *The amalgamated product* $G = G_1 *_{a=x} G_2$ *is not virtually free-by-cyclic.*

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A free-by-cyclic group

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$$\begin{aligned} a_0 &\longmapsto a_0, \\ a_i &\longmapsto a_0^{p_i} a_i a_0^{q_i}, i = 1, 2, ..., n-1, \end{aligned}$$

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- When $n = 3$, $p_i = 0, q_1 = 1, q_2 = 2$, this gives the Gersten group $\langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = ba, tct^{-1} = ca^2 \rangle$.

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- When $p_i = q_i$, this is the group considered by Lyman.

Theorem 3

Let $G = G(\{p_i, q_i\}_{i=1}^{n-1}) = F_n \rtimes_\phi \mathbb{Z}$. *The following are equivalent:*

i) *G is virtually special;*

ii) *G is* $\mathrm{CAT}(0)$*;*

iii) *either*

1) $p_i = -q_i$ *for each* $i = 1, 2, ..., n-1$ *or*
2) $p_s \neq -q_s$ *for some* s *and*

$$q_i(q_i + p_s - q_s) = p_i(p_i - p_s + q_s)$$

for each $i = 1, 2, ..., n-1$.

- The example of Lyman is virtually special, but not cocompact special.

Graph of abelian groups

Theorem 4

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is a cyclic-subgroup-separable tubular group that does not have the property (VRC).

- Key observation: Let G be a (VRC) group. For any infinite-order element $g \in G$, there is an Euclidean space E^k on which G acts isometrically with g acting as a nontrivial translation on some geodesic line.

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- Prove the element a cannot act in this way, by the flat torus theorem.

Amalgamation of free-by-cyclic groups

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- Let G be a free-by-cyclic group and $a \in G$ is not a virtual retractor (i.e. $\exists H <_{f.i.} G$ containing a as a retractor). Then the amalgamated product $G *_{a=t} (F_2 \times \mathbb{Z})$ (where t is a generator of $\mathbb{Z}$) is not virtually free-by-cyclic.

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- In the Gersten group, a is not a virtual retractor.

Tubular groups

- Let Γ be a graph. For each vertex $v \in \Gamma$, put a torus $X_v = T^2$. For each edge $e = [v, w]$, attach a tube $S^1 \times [-1, 1]$ along π_1-injective loops in T_v, T_w.

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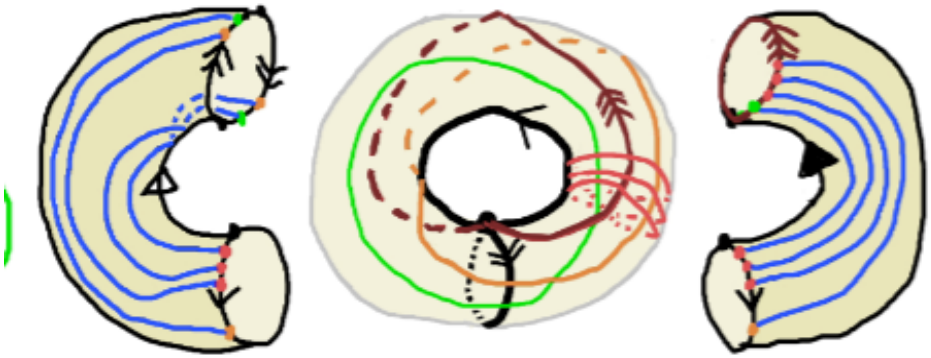
- Let $X = \cup_{S^1 \times [-1,1]} T^2$ be a space obtained from disjoint union of tori T^2 attached by tubes $S^1 \times [-1, 1]$ along π_1-injective loops, called tubular space.



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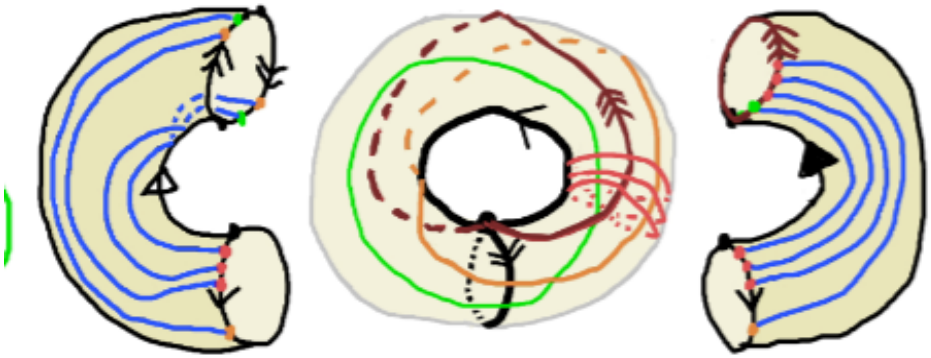

- In other words, a tubular group $G(\Gamma)$ is a graph of groups whose vertex groups are $\mathbb{Z}^2$ and edges groups are $\mathbb{Z}$.

- Note that the semi-direct product $G(\{p_i, q_i\}_{i=1}^{n-1}) = F_n \rtimes_\phi \mathbb{Z}$ has a presentation:

$$\langle a_0, a_1, ..., a_{n-1}, t : ta_0t^{-1} = a_0, ta_it^{-1} = a_0^{p_i}a_ia_0^{q_i}, i = 1, 2, ..., n-1\rangle.$$

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- Relator $ta_it^{-1} = a_0^{p_i}a_ia_0^{q_i}$ is equivalent to $a_it^{-1}a_0^{-q_i}a_i^{-1} = t^{-1}a_0^{p_i}$. Therefore, $G = G(\{p_i, q_i\}_{i=1}^{n-1})$ is a tubular group.

Virtual specialness of tubular groups

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- Let $G = G(\Gamma)$ be a tubular group, $X(\Gamma)$ be a tubular space. For each vertex $v \in \Gamma$, choose a set S_v of immersed loops, such that

$$\sum_{s \in S_v} \#[s, e_+] = \sum_{s \in S_w} \#[s, e_-]$$

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- [Wise] a tubular group $G(\Gamma)$ acts freely on a CAT(0) cubical complex (CCC) if and only there exists an equitable set.
- Caution: The Baumslag-Solitar group $\langle a, t : ta^p t^{-1} = a^q \rangle$ for $p \neq \pm q$ cannot act freely on a CCC.

Virtually specialness of tubular groups

- [Wise] A tubular group is virtually cocompact special if and only only each vertex space has at most two parrellism class of edges, and the group doesn't contain the Baumslag-Solitar groups $\langle a, t : ta^p t^{-1} = a^q \rangle$ for $p \neq \pm q$.

- [Woodhause] The tubular group is virtually special if and only if it acts freely on a finite-dimensional CAT(0) cubical complex if and only if each immersed wall is "non-dialated".

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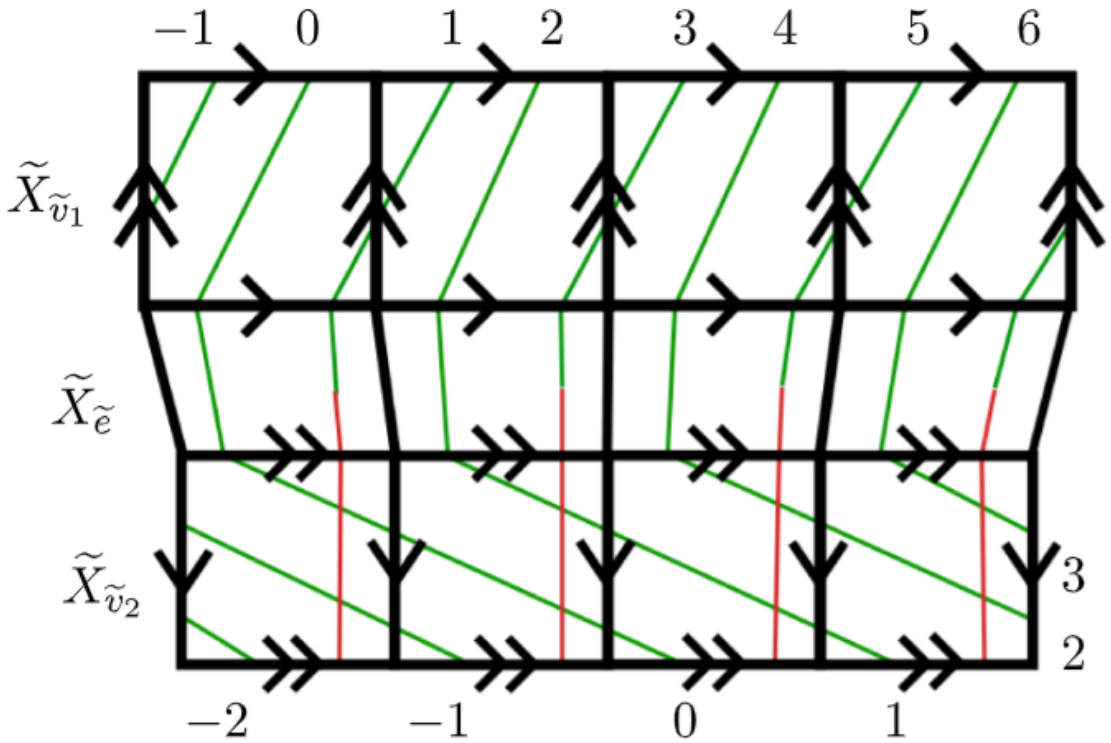
- For example, $G = \langle a, b, t : ab = ba, tat^{-1} = b \rangle$ has equitable set $S = \{a, ab^2\}$. The wall is dialated.



Theorem 6

Let $G = \langle \mathbb{Z}^2, s_i | s_i v_i s_i^{-1} = w_i, i = 1, 2, ..., k \rangle$ *be a* $\mathrm{CAT}(0)$ *free-by-cyclic tubular group. Then* G *is virtually special.*

Idea of the proof: construct non-dialated immersed walls.

Let G be a CAT(0) free-by-cyclic group. Is every finitely generated subgroup of G again CAT(0)?

Thank you for your attention!