Existence of SC^p-metrics on projective-plane-like manifolds

Su Yang

Chinese Academy of Sciences

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Projective plane-like manifolds

A closed smooth manifold M looks like a projective plane if

$$\pi_1(M) = 0, \quad H_*(M) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Example. $M = \mathbb{C}\mathrm{P}^2, \mathbb{H}\mathrm{P}^2, \mathbb{O}\mathrm{P}^2$

By the minimal cell structure theorem $M \simeq S^m \cup_f D^{2m}$

$[f] \in \pi_{2m-1}(S^m)$ has Hopf invariant 1 $\Rightarrow$ $m = 2, 4, 8$.

classification

- $\mathbb{C}\mathrm{P}^2$-like manifolds: $M \cong \mathbb{C}\mathrm{P}^2$ (homeomorphic) by Freedman's classification of 1-connected 4-manifolds.

- $\mathbb{H}\mathrm{P}^2$- and $\mathbb{O}\mathrm{P}^2$-like manifolds:

1. a diffeomorphism classification up to connected sum with homotopy spheres by Eells and Kuiper.
2. completed to a diffeomorphism classification by Kramer and Stolz.

Theorem (Kramer-Stolz)

- *The diffeomorphism class of an $\mathbb{H}\mathrm{P}^2$-like manifold is determined by*

$$p_1(M_t) = 2(1+2t), \quad t \equiv 0, 7, 48, 55 \pmod{56}.$$

- *The diffeomorphism class of an $\mathbb{O}\mathrm{P}^2$-like manifold is determined by*

$$p_2(M_t) = 6(1+2t), \quad t \equiv 0, 127, 16128, 16255 \pmod{16256}.$$

Example. $M_0 = \mathbb{H}\mathrm{P}^2$ or $\mathbb{O}\mathrm{P}^2$.

A complete numerical invariant is the $\hat{A}$-genus

$$\hat{A}(M_t^8) = -\frac{t(t+1)}{7 \cdot 8}, \quad \hat{A}(M_t^{16}) = -\frac{t(t+1)}{127 \cdot 128}$$

SC^p-Riemannian metrics

A Riemannian structure (M, g) is called an SC^p-structure if all geodesics issued from p are simply closed periodic with the same length.

Example. S^n, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, $\mathbb{H}\mathrm{P}^n$, $\mathbb{O}\mathrm{P}^2$

Bott-Samelson: if M carries an SC^p-structure, then

$$\pi_1(M) \cong 0 \ \text{ or } \ \mathbb{Z}/2, \quad H^*(M;\mathbb{Z}) \cong H^*(\mathrm{CROSS};\mathbb{Z})$$

where CROSS = compact rank one symmetric space.

Main result

Example

1. **Bérard-Bergery**: some exotic 10-sphere Σ^{10} admits an SC^p-structure.

2. **Tang-Zhang**: all $\mathbb{H}\mathrm{P}^2$-like manifolds admit SC^p-structures.

Theorem (Martin Kerin, Qian Chao, S.)

Every $\mathbb{O}\mathrm{P}^2$-like manifold M admits an SC^p-structure.

geometric construction

- M an $\mathbb{O}\mathrm{P}^2$-like manifold, then $\pi_1(M) = 0 \;\; i \leq 7, \;\; \pi_8(M) \cong \mathbb{Z}$
- $M = D(\xi) \cup_f D^{16}$, $\xi =$ the normal bundle of $S^8 \subset M$.
- rank 8 vector bundles over S^8 are classified by $\pi_7 SO(8)$,

$$(e, p_2) \colon \pi_7 SO(8) \to \mathbb{Z} \oplus \mathbb{Z}$$

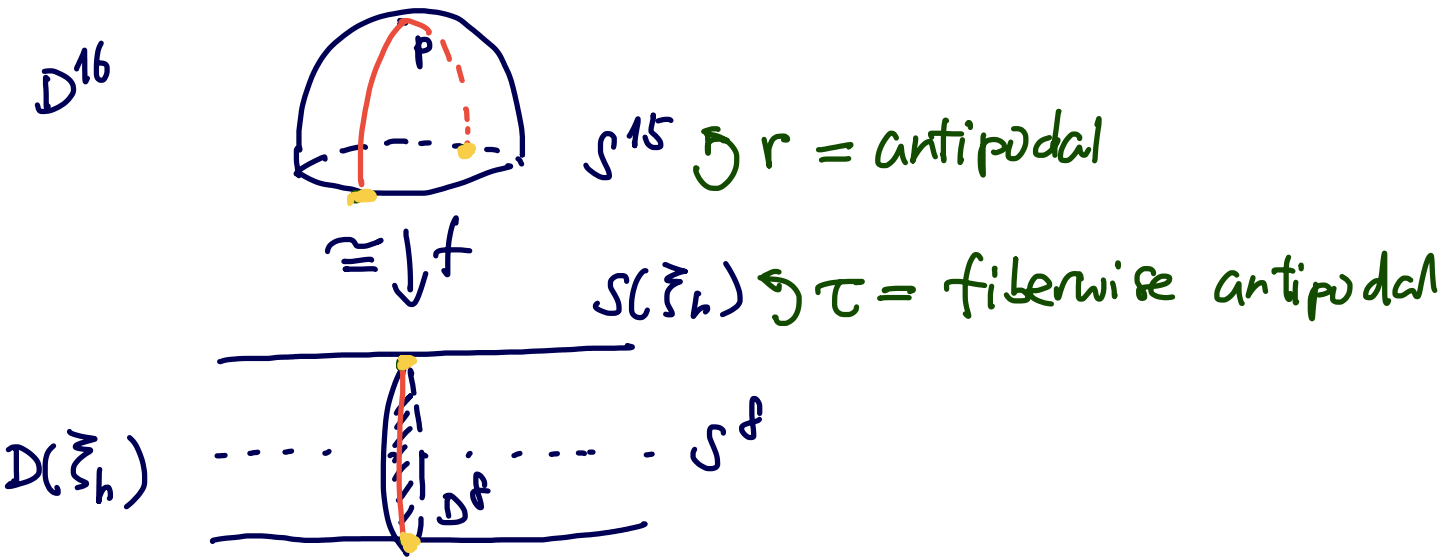
- $S(\xi) = \partial D(\xi) \cong S^{15} \Rightarrow e(\xi) = u \in H^8(S^8)$ a generator
- such bundles are parametrized by $h \in \mathbb{Z}$

$$e(\xi_h) = u, \;\; p_2(\xi_h) = 6(2h-1)u$$

- $\mu(S(\xi_h)) = \frac{h(h-1)}{2^7 \cdot 127} = 0 \pmod{\mathbb{Z}} \Rightarrow h \equiv 0 \text{ or } 1 \pmod{128}$

construction of SC^p-metrics

Bénard-Bergery: For $M_h = D(\xi_h) \cup_f D^{16}$, if $f\colon S^{15} \xrightarrow{\cong} S(\xi_h)$ is $\mathbb{Z}/2$-equivariant, then M_h admits an SC^p-structure.



Conclusion: if $\bar{f}\colon \mathbb{R}\mathrm{P}^{15} \to Q_h = S(\xi_h)/\tau$ is a diffeomorphism, then M_h admits an SC^p-structure.

approaches to the proof

Theorem

The quotient manifold Q_h is diffeomorphic to $\mathbb{R}\mathrm{P}^{15}$.

Approach 1: since $Q_h \simeq \mathbb{R}\mathrm{P}^{15}$, compute with the surgery exact sequence

$$L_{16}(\mathbb{Z}[\mathbb{Z}/2]) \xrightarrow{\omega} \mathcal{S}^{\mathrm{DIFF}}(\mathbb{R}\mathrm{P}^{15}) \xrightarrow{\eta} [\mathbb{R}\mathrm{P}^{15}, G/O] \to L_{15}(\mathbb{Z}[\mathbb{Z}/2])$$

1. show that $\eta(Q_h) = 0$ (accomplished by AI)
2. understand the action of $L_{16}(\mathbb{Z}[\mathbb{Z}/2])$

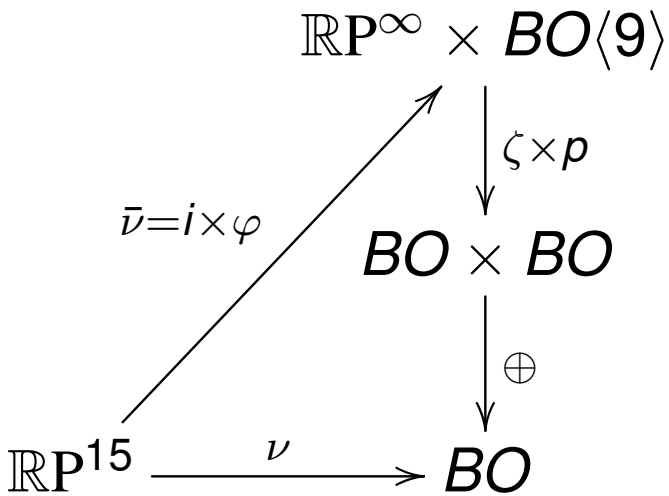
Approach 2: modified surgery replaces the first step by a bordism computation

normal 7-type of $\mathbb{R}\mathrm{P}^{15}$

- $T\mathbb{R}\mathrm{P}^{15} \oplus \underline{\mathbb{R}} \cong \mathrm{Hom}(L, \underline{\mathbb{R}^{16}}) \cong 16L$
- $BO\langle n\rangle \to BO$ the $(n-1)$-connected cover of BO

$$BO\langle 4\rangle = B\,\mathrm{Spin},\ BO\langle 8\rangle = B\,\mathrm{String},\ \ BO\langle 9\rangle = B\,\mathrm{Fivebrane}$$

- the normal Gauss map $\nu\colon \mathbb{R}\mathrm{P}^{15} \to BO$ has an 8-connected lift $\bar{\nu}$



where $\zeta = -16L$, φ is a $BO\langle 9\rangle$-structure on the trivial bundle over $\mathbb{R}\mathrm{P}^{15}$

normal 7-type of Q_h

- $Q_h = S(\xi_h)/\tau = \mathbb{P}(\xi_h)$ the projective bundle of ξ_h, then

$$TQ_h \oplus \underline{\mathbb{R}} = \pi^*(TS^8) \oplus \lambda \otimes \pi^*(\xi_h)$$

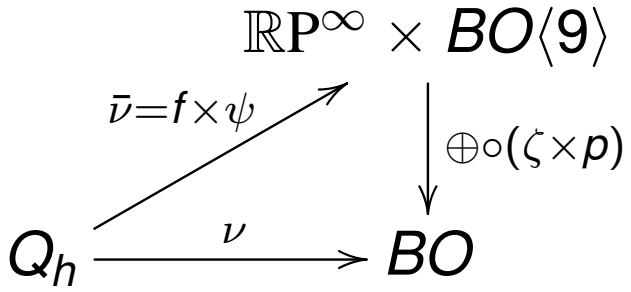
  $\lambda =$ the tautological line bundle over $\mathbb{P}(\xi_h)$

- $f\colon Q_h \to \mathbb{R}\mathrm{P}^\infty$ the classifying map of π_1, then

$$w_i(\nu_{Q_h} - f^*(\zeta)) = 0, \quad i = 1, 2, 4, 8$$

  the stable bundle $\nu_{Q_h} - f^*(\zeta)$ admits $BO\langle 9\rangle$-structures

- the normal Gauss map $\nu\colon Q_h \to BO$ has an 8-connected lift $\bar{\nu}$



bordism class

Proposition

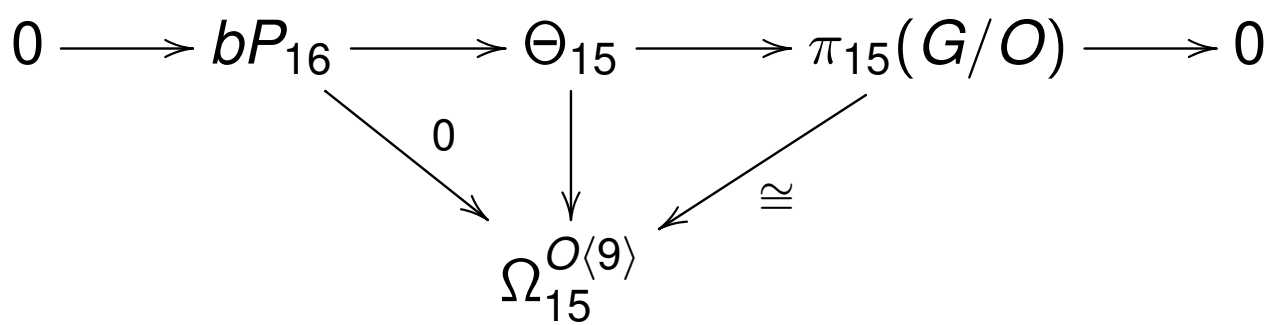
$[\mathbb{R}\mathrm{P}^{15}, \bar{\nu}] = [Q_h, \bar{\nu}] \in \Omega_{15}^{O\langle 9\rangle}(\mathbb{R}\mathrm{P}^{\infty}; \zeta)$

Proof.

- **He-Rajan**: $\exists\, \mathbb{R}\mathrm{P}^{13} \cong N \subset Q_h$, normal bundle $= 2L$
- $T =$ tubular neighborhood of $N \subset Q_h$, $C = Q_h - T$;
  $T_0 =$ tubular neighborhood of $\mathbb{R}\mathrm{P}^{13} \subset \mathbb{R}\mathrm{P}^{15}$, $C_0 = \mathbb{R}\mathrm{P}^{15} - T_0$
  $C_0 \cong S^1 \times D^{14}$, $C \simeq S^1$, normal 7-type $= S^1 \times BO\langle 9\rangle$
- $$\Omega_{15}^{O\langle 9\rangle}(S^1) \to \Omega_{15}^{O\langle 9\rangle}(\mathbb{R}\mathrm{P}^{\infty}; \zeta) \to \Omega_{15}^{O\langle 9\rangle}(S^{\infty}) = \Omega_{15}^{O\langle 9\rangle}$$
  $$[C \cup_{\partial} (-C_0), ?] \mapsto [\mathbb{R}\mathrm{P}^{15}, \bar{\nu}] - [Q_h, \bar{\nu}] \mapsto [S^{15}] - [S^{15}] = 0$$
  note that there is a unique $BO\langle 9\rangle$-structure on S^{15}



- $$\Omega_{15}^{O\langle 9\rangle}(S^1) \cong \Omega_{15}^{O\langle 9\rangle} \oplus \Omega_{14}^{O\langle 9\rangle} \xrightarrow{(2\cdot,\eta)} \Omega_{15}^{O\langle 9\rangle}$$
$$[C \cup (-C_0), ?] = (a, b) \mapsto (2a, \eta \cdot b) = 0$$
where $\eta \in \Omega_1^{\mathrm{fr}} \cong \pi_1^S \cong \mathbb{Z}/2$ the generator
- surgery theory



therefore $\Omega_{15}^{O\langle 9\rangle} \cong \pi_{15}(G/O) \cong \mathbb{Z}/2\{\eta \cdot \kappa\}$
$\kappa =$ the generator of $\pi_{14}(G/O) \cong \Theta_{14} \cong \Omega_{14}^{O\langle 9\rangle} \cong \mathbb{Z}/2$

- taking $O\langle 9\rangle$-bordism class is a homomorphism

$$\pi_7 SO(8) \to \Omega_{15}^{O\langle 9\rangle}, \quad [\xi] \mapsto [S(\xi)]$$

- $[\xi_h] = h[\xi_1] + (1-h)[\xi_0]$, $S(\xi_0) = S(\xi_1) = \mathbb{R}\mathrm{P}^{15}$
- $[Q_h, ?] = [\mathbb{R}\mathrm{P}^{15}, ?] \in \Omega_{15}^{O\langle 9\rangle} \cong \mathbb{Z}/2$

surgery obstruction

$[Q_h, \bar{\nu}]$ and $[\mathbb{R}\mathrm{P}^{15}, \bar{\nu}]$ are cobordant by W, surgery obstruction

$$\theta(W) \in L_{16}(\mathbb{Z}[\mathbb{Z}/2]) \cong L_{16}(\mathbb{Z}) \oplus \widetilde{L}_{16}(\mathbb{Z}[\mathbb{Z}/2]) \cong \mathbb{Z} \oplus \mathbb{Z}$$

1. the summand $L_{16}(\mathbb{Z})$ corresponds to connected sum with Σ^{15}, detected by the Eells-Kuiper μ-invariants

2. the summand $\widetilde{L}_{16}(\mathbb{Z}[\mathbb{Z}/2])$ is detected by the Browder-Liversay invariant

Browder-Liversay invariant

- $\tau\colon \Sigma^{15} \to \Sigma^{15}$ a free involution, there exist characteristic submanifolds $W^{14} \subset \Sigma$.
- perform surgery such that W is 6-connected, denote $K = \mathrm{Ker}\,(H_7(W) \to H_7(A))$, symmetric bilinear form

$$B\colon K \times K \to \mathbb{Z}, \quad (x, y) \mapsto \langle x \cup \tau_*(y), [W] \rangle$$

- the Browder-Liversay invariant of $(\Sigma, \tau) = \mathrm{sign}(B)$

Shimada: $\exists$ a $\mathbb{Z}/2$-equivariant embedded $S^{14} \subset S(\xi_h)$, i.e., $W = S^{14}$. Therefore the Browder-Liversay invariant vanishes.

Eells-Kuiper μ-invariants

- Atiyah-Singer index theorems for M^{16}

1. $\operatorname{sign}(M) = \operatorname{ind}(B) = \langle L_4(p_1(M), \cdots, p_4(M)), [M] \rangle$
2. $\operatorname{ind}(D^+) = \langle \hat{A}_4(p_1(M), \cdots, p_4(M)), [M] \rangle$, M spin

- X^{15} $\mathbb{Q}$-homology sphere with a spin structur s_0, with $(X, s_0) = \partial(W^{16}, s)$ a spin boundary. For $i = 1, 2, 3$

$$H^{4i}(W, X; \mathbb{Q}) \xrightarrow{\cong} H^{4i}(W; \mathbb{Q}), \quad \overline{p}_i(W) \mapsto p_i(W)$$

- denote $F(p_1, p_2, p_3) = \hat{A}(p_1, \cdots, p_4) - t_4 L_4(p_1, \cdots, p_4)$

$$\mu(X, s_0) = \langle F_4(\overline{p}_1(W), \cdots, \overline{p}_3(W)), [W, X] \rangle - t_4 \operatorname{sign}(W) \in \mathbb{Q}/\mathbb{Z}$$

Examples

1. μ detects 7-dimensional homotopy spheres

$$\mu\colon \Theta_7 \xrightarrow{\cong} \mathbb{Z}/28$$

2. for $X = S(\xi_h) = N_h$, we use $W = D(\xi_h)$ and get

$$\mu(S(\xi_h)) = \frac{h(h-1)}{2^7 \cdot 127} \pmod{\mathbb{Z}}$$

intrinsic definition of μ

- Atiyah-Patodi-Singer index theorems for M^{16} with $\partial M = N$

$$\begin{aligned} \operatorname{sign}(M) &= \int_M L_k(p_1(M), \cdots, p_4(M)) - \eta(B_N) \\ \operatorname{ind}(D_M^+) &= \int_M \hat{A}(p_1(M), \cdots, p_4(M)) - \frac{1}{2}(\eta(D_N^+) + \dim\ker(D_N^+)) \end{aligned}$$

where for an elliptic self-adjoint operator A

$$\eta_A(s) = \sum_{\lambda \neq 0} \frac{\operatorname{sign}(\lambda)}{|\lambda|^s}, \quad \eta(A) = \eta_A(0) \quad \text{(spectral asymmetry)}$$

- an intrinsic formula for μ

$$\mu(X) = \frac{1}{2}(\eta(D_X^+) + \dim\ker D_X^+) - t_4\eta(B_X) - \int_X \alpha_4(X) \pmod{\mathbb{Z}}$$

equivariant index theorems

- $W_h = D(\xi_h)$, $N_h = S(\xi_h)$, $Q_h = N_h/\tau$
- antipodal involution $\tau\colon W_h \to W_h$, fixed-point set $W_h^\tau = S^8$, normal bundle $\nu(S^8 \subset W_h) = \xi_h$.
- Atiyah-Bott equivariant index theorems

$$
\begin{aligned}
\operatorname{ind}(D^+_{W_h}) &= \int_{W_h} \hat{A} - \frac{1}{2}(\eta(D_{N_h}) + h(D_{N_h})) \\
\operatorname{ind}_\tau(D^+_{W_h}) &= a_{\text{spin}}(S^8) - \frac{1}{2}(\eta_\tau(D_{N_h}) + h_\tau(D_{N_h})) \\
\operatorname{sign}(W_h) &= \int_{W_h} L - \eta(B_{N_h}) \\
\operatorname{sign}(W_h, \tau) &= a_{\text{sign}}(S^8) - \eta_\tau(B_{N_h})
\end{aligned}
$$

computation of $\mu(Q_h)$

equivariant formula for η-invariants

$$\begin{aligned}
\eta(D_{Q_h}) &= \frac{1}{2}(\eta(D_{N_h}) + \eta_\tau(D_{N_h})) \\
h(D_{Q_h}) &= \frac{1}{2}(h(D_{N_h}) + h_\tau(D_{N_h})) \\
\eta(B_{Q_h}) &= \frac{1}{2}(\eta(B_{N_h}) + \eta_\tau(B_{N_h}))
\end{aligned}$$

$$\begin{aligned}
\mu(Q_h) &= \frac{1}{2}(\eta(D^+_{Q_h}) + \dim\ker D^+_{Q_h}) - t_4\eta(B_{Q_h}) - \int_{Q_h}\alpha_4(Q_h) \\
&= \frac{1}{4}(\eta(D_{N_h}) + h(D_{N_h}) + \eta_\tau(D_{N_h}) + h_\tau(D_{N_h})) \\
&\quad -\frac{1}{2}t_4(\eta(B_{N_h}) + \eta_\tau(B_{N_h})) - \frac{1}{2}\int_{N_h}\alpha_4(N_h) \\
&= \frac{1}{2}\mu(N_h) + \frac{1}{2}a_{\mathrm{spin}}(S^8) + \frac{1}{2}t_4(\mathrm{sign}(W_h,\tau) - a_{\mathrm{sign}}(S^8))
\end{aligned}$$

Conclusion: If $M_h \cong S^{15}$, then

$$\{\mu_+(Q_h), \mu_-(Q_h)\} = \{\frac{1}{512}, -\frac{1}{512}\} = \{\mu_+(\mathbb{R}\mathbf{P}^{15}), \mu_-(\mathbb{R}\mathbf{P}^{15})\}$$

Thanks!