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Linear Circle Quotients of Products of Odd-Dimensional Spheres

Classification and Geometric Applications

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joint work with Yang Su

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Background: circle quotients and curvature

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- Circle quotients of compact Lie groups and products of spheres provide many of the classical examples of manifolds.
- These circle actions are encoded by integer weights; arithmetic data can therefore enter the topology and smooth classification.
- Three guiding families are:
  - Aloff–Wallach spaces in dimension 7;
  - Bazaikin biquotients in dimension 13;
  - Einstein metrics on circle bundles.

A model example: Aloff–Wallach spaces

For coprime integers k, l satisfying $kl(k+l) \neq 0$, embed S^1 into $\mathrm{SU}(3)$ by

$$i_{k,l}(z) = \mathrm{diag}(z^k, z^l, z^{-(k+l)}), \qquad z \in S^1.$$

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$$W_{k,l} = \mathrm{SU}(3)/i_{k,l}(S^1), \qquad \dim W_{k,l} = 7.$$

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Put

$$N(k,l) = k^2 + kl + l^2, \qquad P(k,l) = kl(k+l).$$

Kreck–Stolz classification (oriented category)

For coprime pairs (k,l) and (k',l'), with $N = N(k,l) = N(k',l')$:

$$W_{k,l} \cong_{\mathrm{homeo}} W_{k',l'} \iff P(k,l) \equiv P(k',l') \pmod{24N},$$
$$W_{k,l} \cong_{\mathrm{Diff}} W_{k',l'} \iff P(k,l) \equiv P(k',l') \pmod{96 \cdot 7^{\delta_7(N)} N},$$

where $\delta_7(N) = 0$ if $7 \mid N$ and 1 otherwise.

Kreck–Stolz, J. Differential Geom. 33 (1991), 465–486.

Another model: Bazaikin spaces

Let $q = (q_0, \ldots, q_5) \in \mathbb{Z}^6$ satisfy $\sum_{i=0}^{5} q_i = 0$. Define

$$B_q = \mathrm{diag}(z^{q_0}, \ldots, z^{q_5}) \backslash \mathrm{SU}(6) / \mathrm{Sp}(3).$$

For the quotient to be a manifold, the effective action is free; in the Florit–Ziller convention this is equivalent to all q_i odd and

$$\gcd(q_{\tau(1)} + q_{\tau(2)}, q_{\tau(3)} + q_{\tau(4)}) = 2$$

for every permutation $\tau \in S_6$.

Bazaikin spaces: characteristic data

Write $\sigma_j = \sigma_j(q)$ for the elementary symmetric polynomials and let s be defined by $8s = |\sigma_3|$. Then

$$H^6(B_q;\mathbb{Z}) = H^8(B_q;\mathbb{Z}) = \mathbb{Z}_s.$$

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The characteristic data computed by Florit–Ziller include

$$p_1(B_q) = -\sigma_2 = \frac{1}{2}\sum_{i=0}^{5} q_i^2, \qquad 8p_2(B_q) = 3p_1(B_q)^2 - \sigma_4 \quad \text{in } \mathbb{Z}_s,$$

and the linking form is represented by

$$\mathrm{lk}(B_q) = \pm 32\,\sigma_5^{-1} \in \mathbb{Z}_s.$$

Florit–Ziller, J. Eur. Math. Soc. 11 (2009), 189–205.

Einstein metrics on circle bundles

Let α_1, α_2 generate $H^2(\mathbb{CP}^n \times \mathbb{CP}^n; \mathbb{Z})$. For coprime positive integers k, l, consider the principal circle bundle

$$S^1 \longrightarrow M^{(n)}_{k,l} \longrightarrow \mathbb{CP}^n \times \mathbb{CP}^n, \qquad e = k\alpha_1 + l\alpha_2.$$

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Its total space is the repeated-weight quotient

$$M^{(n)}_{k,l} = \frac{S^{2n+1} \times S^{2n+1}}{S^1}, \qquad z \cdot (u, v) = (z^k u, z^l v).$$

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What this motivates

Classify $M^{(2)}_{k,l}$ and $M^{(3)}_{k,l}$.

Wang–Ziller, J. Differential Geom. 31 (1990), 215–248.

General linear circle quotients

Choose weights

$$a = (a_1, \ldots, a_{n+1}), \qquad b = (b_1, \ldots, b_{n+1}) \in \mathbb{Z}^{n+1}.$$

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$$z \cdot ((u_1, \ldots, u_{n+1}), (v_1, \ldots, v_{n+1})) = ((z^{a_1} u_1, \ldots, z^{a_{n+1}} u_{n+1}), \\ (z^{b_1} v_1, \ldots, z^{b_{n+1}} v_{n+1})).$$

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If the action is free, set

$$M_{a,b} := (S^{2n+1} \times S^{2n+1})/S^1.$$

Its dimension is

$$\dim M_{a,b} = 2(2n+1) - 1 = 4n + 1.$$

Freeness, cohomology, and curvature

Theorem (Freeness criterion)

The action is free if and only if

$$\gcd(|a_i|, |b_j|) = 1 \qquad \text{for all } i, j.$$

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For every free action, $M_{a,b}$ is simply connected and

$$H^*(M_{a,b}; \mathbb{Z}) \cong \mathbb{Z}[x, y]/(x^{n+1}, y^2), \qquad |x| = 2, \quad |y| = 2n + 1.$$

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- All quotients in this family have the same integral cohomology ring.
- The product round metric is invariant under the action.
- O'Neill's formula therefore gives a quotient metric of nonnegative sectional curvature.

The bundle $\Gamma_{a,b}$ and the stable tangent bundle

Let $L \to \mathbb{CP}^\infty$ be the universal complex line bundle. The weight vector defines a representation

$$\rho_{a,b} : S^1 \longrightarrow \mathrm{U}(2n+2), \qquad z \longmapsto \mathrm{diag}(z^{a_1}, \ldots, z^{a_{n+1}}, z^{b_1}, \ldots, z^{b_{n+1}}).$$

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The associated complex vector bundle is

$$\boxed{\Gamma_{a,b} = ES^1 \times_{\rho_{a,b}} \mathbb{C}^{2n+2} \cong \bigoplus_{i=1}^{n+1} L^{a_i} \oplus \bigoplus_{j=1}^{n+1} L^{b_j} \longrightarrow \mathbb{CP}^\infty.}$$

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Let $f : M_{a,b} \to \mathbb{CP}^\infty$ classify the principal circle bundle. The tangent-bundle identity is

$$\boxed{TM_{a,b} \oplus \varepsilon^3 \cong f^*(\Gamma_{a,b,\mathbb{R}}).}$$

Characteristic classes and the extra KO-invariant

If $x = f^* c_1(L)$, then

$$p(TM_{a,b}) = \prod_{i=1}^{n+1}(1 + a_i^2 x^2) \prod_{j=1}^{n+1}(1 + b_j^2 x^2).$$

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Thus

$$p_k(M_{a,b}) = \Pi_k(a, b)x^{2k}, \qquad \Pi_k = e_k(a_1^2, \ldots, a_{n+1}^2, b_1^2, \ldots, b_{n+1}^2).$$

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When $n = 4r + 1$, there is one additional stable real KO-torsion coefficient:

$$\Delta_r(a,b) = \sum_i c_{2r+1}(|a_i|) + \sum_j c_{2r+1}(|b_j|) \pmod 2,$$

where

$$c_j(m) = \begin{cases} \frac{m}{j}\binom{m+j-1}{2j-1}, & m \geq j, \\ 0, & m < j. \end{cases}$$

Main theorem: PL homeomorphism classification

Theorem (PL classification)

For any $n \geq 1$, two orbit manifolds $M_{a,b}$ and $M_{a',b'}$ are PL homeomorphic if and only if

$$\Pi_k(a,b) = \Pi_k(a',b') \quad (1 \leq k \leq \lfloor n/2 \rfloor),$$

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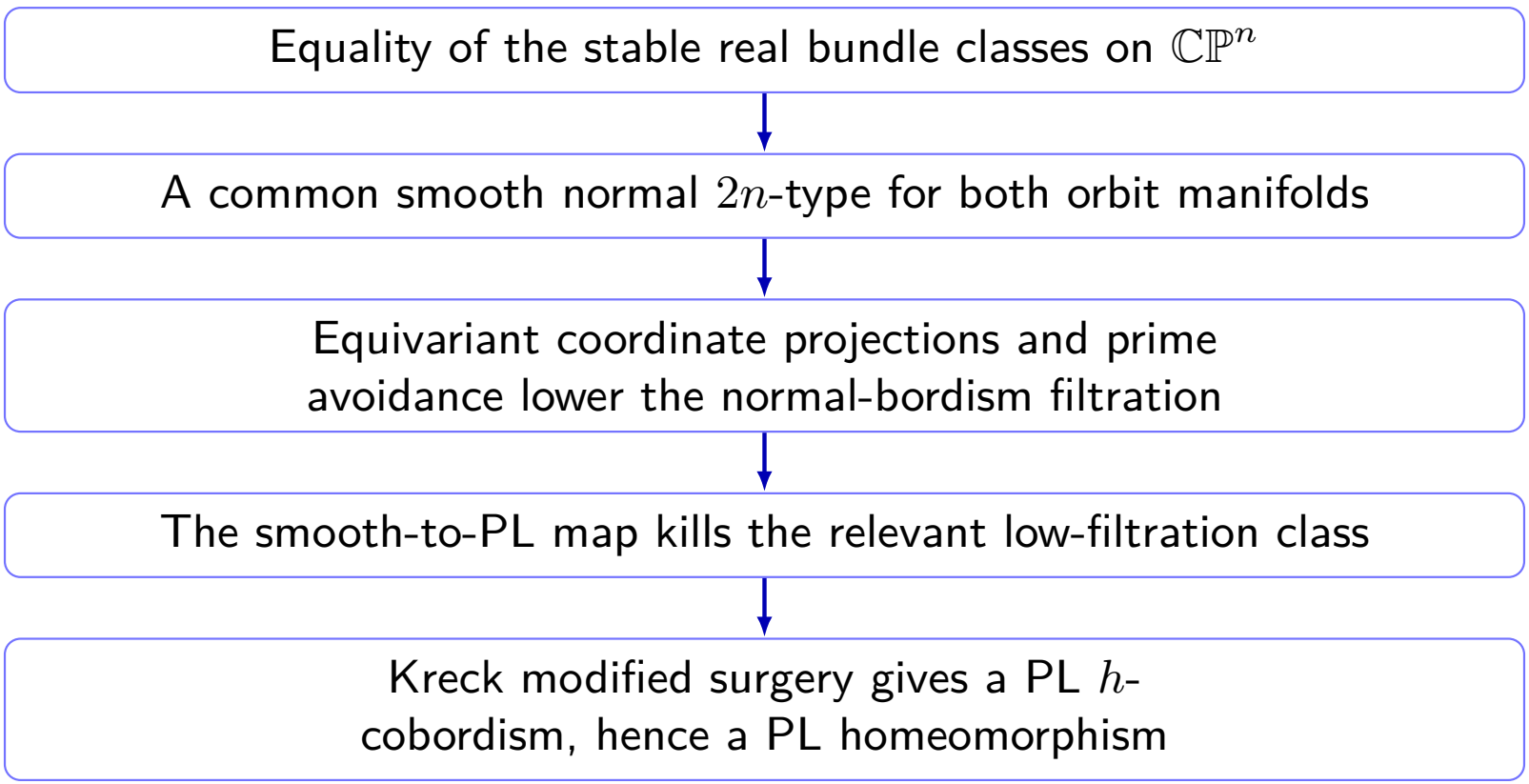
Equivalently, set

$$\gamma_{a,b} = [j_n^* \Gamma_{a,b,\mathbb{R}}] - [\varepsilon^{4n+4}] \in \widetilde{KO}(\mathbb{CP}^n),$$

where $j : \mathbb{CP}^n \xrightarrow{i} M_{a,b} \xrightarrow{f} \mathbb{CP}^\infty$ is the inclusion. Then the PL type is determined precisely by the stable real bundle class $\gamma_{a,b}$ on the sub-complex $\mathbb{CP}^n \subset M_{a,b}$.

PL proof: a common normal type and filtration control



The central idea is a comparison of normal-bordism filtrations; one does not need to solve the homotopy classification first.

The filtration comparison in more detail

The smooth normal-bordism group is computed by an Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(M\xi; \Omega_t^{O\langle 2n+2\rangle}) \implies \Omega_{s+t}^{O\langle 2n+2\rangle}(M\xi).$$

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Fix the total degree $m = s + t$. The s-th filtration subgroup is

$$F_{s,m-s} = \operatorname{im}\left(\Omega_m^{O\langle 2n+2\rangle}(M\xi^{(s)}) \longrightarrow \Omega_m^{O\langle 2n+2\rangle}(M\xi)\right).$$

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$$F_{s,m-s}/F_{s-1,m-s+1} \cong E^\infty_{s,m-s}.$$

Filtration statement

The canonical normal-smoothing class is forced into the low filtration

$$[M, \bar{\nu}_M] \in F_{2n,\,2n+1}.$$

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The canonical normal-smoothing class is forced into the low filtration

$$[M, \bar{\nu}_M] \in F_{2n,\,2n+1}.$$

- After forgetting the vector-bundle structure and passing from smooth to PL normal bordism, the induced map vanishes on this filtration.
- Modified surgery then produces the desired PL homeomorphism.

PL sphere-bundle realization

A second structural theorem gives a geometric model over a fixed base.

Theorem (Sphere-bundle realization)

For $n \geq 1$, every $M_{a,b}$ is PL homeomorphic to the unit sphere bundle of an oriented real vector bundle

$$\eta_{a,b} \longrightarrow \mathbb{CP}^n, \qquad \operatorname{rank}_{\mathbb{R}}(\eta_{a,b}) = 2n+2 :$$

$$\boxed{M_{a,b} \cong_{\mathrm{PL}} S(\eta_{a,b}).}$$

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The stable bundle relation is

$$[T\mathbb{CP}^n] + [\eta_{a,b}] = [i^* T M_{a,b}] + [\varepsilon^1] \quad \text{in } KO(\mathbb{CP}^n)$$

where $i : \mathbb{CP}^n \to M_{a,b}$ is the inclusion.

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- The sphere-bundle model shows that homotopy classification is governed by the stable spherical fibration associated to $\eta_{a,b}$.

Homotopy classification in dimensions 9 and 13

For $n = 2, 3$, Adams–Walker gives

$$\widetilde{J}(\mathbb{CP}^2) \cong \mathbb{Z}/24, \qquad \widetilde{J}(\mathbb{CP}^3) \cong \mathbb{Z}/24.$$

Put

$$\lambda(a, b) = \sum_{i=1}^{n+1} (a_i^2 + b_i^2) = \Pi_1(a, b).$$

Then the J-criterion reduces to

$$\boxed{M_{a,b} \simeq M_{a',b'} \iff \lambda(a, b) \equiv \lambda(a', b') \pmod{24}.}$$

- $n = 2$: dimension 9;
- $n = 3$: dimension 13.

Adams–Walker, Proc. Cambridge Philos. Soc. 61 (1965), 81–103.

Smooth classification in dimension 13

The characteristic data include

$$p_1(M_{a,b}) = \lambda x^2, \qquad w_2(M_{a,b}) \equiv \lambda x \pmod 2.$$

Theorem

Let $M_{a,b}$ and $M_{a',b'}$ be the quotients of $S^7 \times S^7$ by two free diagonal linear S^1-actions. Suppose λ and λ' are both odd or both divisible by 4. Then

$$M_{a,b} \cong_{\mathrm{Diff}} M_{a',b'} \quad \Longleftrightarrow \quad \lambda(a,b) = \lambda(a',b').$$

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- If $4 \mid \lambda$, we use the surgery exact sequence.

Dimension 13: surgery and the pinch-map correction

For a 13-manifold, the relevant part of the surgery sequence is

$$[\Sigma M, G/O] \longrightarrow L_{14}(e) \cong \mathbb{Z}/2 \longrightarrow \mathcal{S}^{\mathrm{Diff}}(M) \xrightarrow{\eta^{\mathrm{Diff}}} [M, G/O] \longrightarrow L_{13}(e) = 0.$$

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Let $i : \mathbb{CP}^3 \hookrightarrow M$ be the inclusion, let $H : S^7 \to \mathbb{CP}^3$ be the Hopf map, and choose $\nu_7^2 : S^{13} \to S^7$ representing $\nu^2 \in \pi_6^S$. Pinch a disk disjoint from $i(\mathbb{CP}^3)$:

$$c : M \longrightarrow M \vee S^{13}, \qquad \sigma = i \circ H \circ \nu_7^2,$$

$$f_{\nu^2}^M = (\mathrm{id}_M \vee \sigma) \circ c.$$

Smooth classification in dimension 9

The characteristic data include

$$p_1(M_{a,b}) = \lambda x^2, \qquad w_2(M_{a,b}) \equiv \lambda x \pmod 2.$$

Theorem

Let $M_{a,b}$ and $M_{a',b'}$ be the quotients of $S^5 \times S^5$. Suppose both manifolds are non-spin, so that λ and λ' are both odd. Then

$$M_{a,b} \cong_{\text{Diff}} M_{a',b'} \quad \Longleftrightarrow \quad \lambda(a,b) = \lambda(a',b').$$

Application I: homogeneous Einstein metrics

Let $N_m = p_1 \cdots p_m$, where the p_i are distinct primes with $p_i \equiv 1 \pmod 4$. There are exactly 2^{m-1} primitive representations

$$N_m = k^2 + l^2, \qquad 0 < k < l, \quad \gcd(k, l) = 1.$$

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For repeated weights, the corresponding orbit manifolds have

$$\lambda = 3N_m \quad (n = 2,\ \dim = 9,\ \ M^{(2)}_{k,l}), \qquad \lambda = 4N_m \quad (n = 3,\ \dim = 13,\ \ M^{(3)}_{k,l}).$$

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Let $N_m = p_1 \cdots p_m$, where the p_i are distinct primes with $p_i \equiv 1 \pmod 4$. There are exactly 2^{m-1} primitive representations

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- Wang–Ziller metrics associated to different pairs (k, l) have distinct unit-volume Einstein constants on $M^{(2)}_{k,l}$ ($M^{(3)}_{k,l}$) for fixed N_m.

Conclusion

Each resulting manifold has at least 2^{m-1} connected components in the moduli space of unit-volume homogeneous Einstein metrics, even when the transitive Lie group is not fixed in advance.

Application II: regular Sasakian structures

For $k, l > 0$, the Euler class

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is represented by an integral Kähler form on $\mathbb{CP}^n \times \mathbb{CP}^n$.

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- Distinct representations $N_m = k^2 + l^2$ with $0 < k < l$ have distinct values of $l - k$.

Hence their contact structures are pairwise non-contactomorphic after transport to the same smooth manifold.

Conclusion

The same 9- and 13-dimensional manifolds carry at least 2^{m-1} regular Sasakian structures with pairwise non-contactomorphic.

Thanks for your attention!