Reduced characteristic number criteria for equivariant bordisms

(Joint work with Runze Chen and Leqi Yang)

Zhi Lü
Fudan University

Greater Bay Area Topology Conference 2026

Topology of Manifolds
& Topological Data Analysis

Outline

- Equivariant geometric bordism
- Motivation and problem
- Main results
- Methods and proofs
- Application to Kosniowski's conjecture

§1 Equivariant Geometric Bordism

- In the early 1960's, Conner-Floyd[1]:

  Bordism + Transformation groups $\Rightarrow$ Equivariant Bordism

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- **Fundamental Problems**:

  1) Determination of complete invariants for equivariant bordisms.

  2) Determination of ring $\Omega_*^{\mathbb{S},G}$, formed by smooth closed G-manifolds with $\mathbb{S}$ as structure group on tangent bundles.

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Geometric Bordism Ring

- $\Omega_*^{\mathbb{S}} = \bigoplus_n \Omega_n^{\mathbb{S}}$, where $\Omega_n^{\mathbb{S}} = \{[M^n] \mid M^n \text{ is closed smooth with structure group } \mathbb{S}\}$
- Addition: $[M^n] + [N^n] = [M^n \sqcup N^n]$;
- Multiplication: $[M^m] \times [N^n] = [M^m \times N^n]$

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Definitions for equivariant case can be given similarly.

Some Known Results for non-equivariant case

A List

Geometric Bordism	Structure group $\mathbb{S}$	Ring	Complete invariants
unoriented (Thom)	O	$\Omega_*^O \cong \pi_*(MO)$	Stiefel–Whitney numbers
orientable (Wall et al.)	SO	$\Omega_*^{SO} \cong \pi_*(MSO)$	Stiefel–Whitney numbers Pontryagin numbers
unitary (Milnor, Novikov)	U	$\Omega_*^U \cong \pi_*(MU)$	Chern numbers
special unitary (Conner–Floyd et al.)	SU	$\Omega_*^{SU} \cong \pi_*(MSU)$	Chern numbers KO-theory char. numbers
general case	$\mathbb{S}$	$\Omega_*^{\mathbb{S}} \cong \pi_*(M\mathbb{S})$ (Pontryagin-Thom isomorphism)[1]	$\cdots$

[1] R. E. Stong, Notes on cobordism theory, Princeton University Press, 1968

Equivariant Case–Development for complete invariants

Action Group	Structure group $\mathbb{S}$	Complete invariants
$G=\mathbb{Z}_2^k$ (tom Dieck:1971)[1]	O	Equivariant Stiefel–Whitney numbers
$G=T^k\times\mathbb{Z}_m$ (tom Dieck:1974)[2] [3]	U	Equivariant K theoretic Chern numbers
$G=T^k$ (L–Wang:2018)[4]	U	Equivariant cohomology Chern numbers (conjectured by Guillemin, Ginzburg and Karshon)[5]

[1] T. tom Dieck, *Characteristic numbers of G-manifolds. I*, Invent. Math **13** (1971),213–224.

[2] T. tom Dieck, *Characteristic numbers of G-manifolds, II*, J. Pure Appl. Algebra **4** (1974), 31-39.

[3] M. Bix, T. tom Dieck, Characteristic numbers of G-manifolds and multiplicative induction, Trans. Amer. Math. Soc. **235** (1978), 331–343.

[4] Z. Lü and W. Wang, *Equivariant cohomology Chern numbers determine equivariant unitary bordism for torus groups*. Algebr. Geom. Topol. **18** (2018), no. 7, 4143-4160.

[5] V Guillemin, V Ginzburg, Y Karshon, Moment maps, cobordisms, and Hamiltonian group actions, Mathematical Surveys and Monographs 98, Amer. Math. Soc., Providence, RI (2002).

A simple review on the study of $\Omega_*^{\mathbb{S},G}$

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- So, much work is on the study of the module structure and the homotopy description of $\Omega_*^{\mathbb{S},G}$. [3]

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  Remark. Recently, counterexamples have been found[5] [6]

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§2 Motivation and problem

- The equivariant manifolds with isolated fixed points (especially, the case of unitary T^k-manifolds) bear distinctive intrinsic features and rich theoretical profundity, giving rise to numerous theoretical frameworks and research branches, including toric varieties in algebraic geometry, toric geometry, DJ theory, GKM theory and toric topology. [1] [2] [3] [4] [5]

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- In theory, to judge whether a unitary T^k-manifold is an equivariant boundary is to require computing the **complete collection** of equivariant characteristic numbers, imposing **heavy computational demands**.

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- In theory, to judge whether a unitary T^k-manifold is an equivariant boundary is to require computing the **complete collection** of equivariant characteristic numbers, imposing **heavy computational demands**.
  However, **several partial simplifications** have been achieved for unitary T^k-manifolds with isolated fixed points in specific dimensional cases.

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Case-specific results and problem

Let M be a unitary T^k-manifold with isolated fixed points.

- (L-Tan, 2011) For $\dim M = 2k$, only the equivariant cohomology Chern numbers constructed from the first and second equivariant Chern classes are sufficient to determine the equivariant unitary bordism class of M. [1]
- (Wen-Ma, 2019) For $\dim M = 2(k-1)$, the first six equivariant Chern classes can play the same role as the case $\dim M = 2k$. [2]

These case-specific results show that we do not need the entire family of equivariant cohomology Chern numbers to judge whether a unitary T^k-manifold is an equivariant boundary.

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These case-specific results show that we do not need the entire family of equivariant cohomology Chern numbers to judge whether a unitary T^k-manifold is an equivariant boundary. This naturally raises the following problem:

Problem A

Generally, **what minimal collection of equivariant cohomology Chern numbers can fully characterize the equivariant unitary bordism class of M?**

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§3 Main Results–Reduced characteristic number criteria

The first main result [1] is as follows:

Theorem (Chen–L-Yang)

Let M^{2n} be a $2n$-dimensional unitary T^k-manifold with a finite fixed point set M^{T^k}. Then M^{2n} bounds equivariantly if and only if there exists a distinguishing polynomial g such that

$$\langle g^l, [M] \rangle = 0$$

for all integers $l < |M^{T^k}|$, where $[M]$ denotes the fundamental homology class of M.

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- A distinguishing polynomial is a line combination of equivariant Chern classes satisfying local fixed-point conditions.

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- Such polynomials always exist. This yields the above reduced bordism characterization, thus resolving Problem A.

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Real Case

The second main result[1] is as follows:

Theorem (Chen–L-Yang)

Let M^n be an n-dimensional closed smooth $(\mathbb{Z}_2)^k$-manifold with a finite fixed point set $M^{(\mathbb{Z}_2)^k}$. Then M^n bounds equivariantly if and only if

$$\left\langle \left(w_n^{(\mathbb{Z}_2)^k}(TM)\right)^l, [M] \right\rangle = 0$$

for all integers $l \leq \left|M^{(\mathbb{Z}_2)^k}\right|$, where $w_n^{(\mathbb{Z}_2)^k}(TM)$ denotes the top equivariant Stiefel–Whitney class of the tangent bundle TM, and $[M]$ is the mod-2 fundamental homology class of M.

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Remark. In contrast to the complex setting with T^k-actions, the real counterpart equipped with $(\mathbb{Z}_2)^k$-actions possesses much stronger structural properties: the top equivariant Stiefel–Whitney class alone is sufficient.

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§4 Methods and proofs

- The techniques employed throughout our proofs are mainly the **localization formulae** and Vandermonde matrix arguments.

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- The localization formulae used here are
  1. Atiyah–Bott–Berline–Vergne localization formula for unitary T^k-manifolds;[1] [2]
  2. tom Dieck–Kosniowski–Stong localization formula for smooth closed $(\mathbb{Z}_2)^k$-manifolds[3] [4]

  which furnish an effective approach for computing equivariant characteristic numbers from fixed-point data.

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Equivariant cohomology Chern numbers

Let M be a unitary T^k-manifold of dimension $2n$ with isolated fixed points.

- The **total equivariant cohomology Chern class** of TM is defined as

$$c^{T^k}(TM) := c(ET^k \times_{T^k} TM) \in H^*(ET^k \times_{T^k} M) = H^*_{T^k}(M)$$

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- It can formally be written as $c^{T^k}(TM) = \prod_{i=1}^{n}(1 + x_i)$ with $\deg x_j = 2$ according to the splitting principle, so the i-th equivariant cohomology Chern class $c_i^{T^k}(TM)$ is the i-th elementary symmetric function $\sigma_i(x_1, ..., x_n)$.

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- It can formally be written as $c^{T^k}(TM) = \prod_{i=1}^{n}(1 + x_i)$ with $\deg x_j = 2$ according to the splitting principle, so the i-th equivariant cohomology Chern class $c_i^{T^k}(TM)$ is the i-th elementary symmetric function $\sigma_i(x_1, ..., x_n)$.
- The **equivariant cohomology Chern numbers** of M are defined as

$$\langle f(x_1, ..., x_n), [M] \rangle := p_!^{T^k}(f(x_1, ..., x_n)) \in H^*(BT^k) = \mathbb{Z}[t_1, ..., t_k]$$

where $f(x_1, ..., x_n)$: symmetric polynomial function, and
$p_!^{T^k} : H^*_{T^k}(M) \longrightarrow H^{*-2n}(BT^k)$ is the equivariant Gysin map induced by the constant $p : M \longrightarrow \mathrm{pt}$.

Atiyah–Bott–Berline–Vergne localization formula

Take a fixed point $p \in M^{T^k}$.

- The normal bundle ν_p to p in M^{2n} is T_pM with the orientation inherited from M^{2n}.

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- The equivariant Euler class $e^{T^n}(\nu_p) = \varepsilon_p e^{T^n}(T_pM)$, where ε_p is equal to $+1$ if the orientation of ν_p agrees with the complex orientation on the stable tangent bundle TM and -1 otherwise.

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$$c^{T^k}(T_pM) = \prod_{i=1}^{n}(1 + \lambda_{p,i}) \in H^*(BT^k) = \mathbb{Z}[t_1, ..., t_k]$$

so the equivariant Euler class

$$e^{T^n}(T_pM) = c_n^{T^n}(T_pM) = \lambda_{p,1} \cdots \lambda_{p,n} = \sigma_n(\lambda_{p,1}, ..., \lambda_{p,n}).$$

Atiyah–Bott–Berline–Vergne localization formula

Theorem (ABBV localization formula)

Let M^{2n} be a $2n$-dimensional unitary T^k-manifold with a finite fixed point set M^{T^k}. Then

$$\langle f(x_1,...,x_n),[M]\rangle = \sum_{p\in M^{T^k}} \frac{\varepsilon_p i_p^*(f(x_1,...,x_n))}{e^{T^k}(T_pM)} = \sum_{p\in M^{T^k}} \frac{\varepsilon_p f(\lambda_{p,1},...,\lambda_{p,n})}{\sigma_n(\lambda_{p,1},...,\lambda_{p,n})} \in H^*(BT^k)$$

where $f(x_1,...,x_n)$ is a symmetric polynomial (i.e., a polynomial of equivariant Chern classes), and $i_p^*: H_{T^k}^*(M) \longrightarrow H_{T^k}^*(\{p\}) = H^*(BT^k)$ induced by the inclusion $i_p: \{p\} \hookrightarrow M$ maps $f(x_1,...,x_n)$ to $f(\lambda_{p,1},...,\lambda_{p,n})$.

tom Dieck–Kosniowski–Stong localization formula

Theorem (tom Dieck–Kosniowski–Stong localization formula)

Suppose that M^n is an n-dimensional smooth closed $(\mathbb{Z}_2)^k$-manifold with a finite fixed point set $M^{(\mathbb{Z}_2)^k}$. Then for any symmetric polynomial $f(x_1, ..., x_n)$,

$$\langle f(x_1, ..., x_n), [M] \rangle = \sum_{p \in M^{(\mathbb{Z}_2)^k}} \frac{f(\lambda_{p,1}, ..., \lambda_{p,n})}{\sigma_n(\lambda_{p,1}, ..., \lambda_{p,n})} \in H^*(B(\mathbb{Z}_2)^k; \mathbb{Z}_2).$$

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Remark. If $\deg f < \dim M$, then $\langle f(x_1, ..., x_n), [M] \rangle = 0$.

Outline of Proof

Only consider the unitary case.

- Define an equivalence relation $\sim$ on M^{T^k} in such a way:

$$p \sim q \Longleftrightarrow c^{T^k}(T_pM) = c^{T^k}(T_qM).$$

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- A symmetric polynomial $g(x_1, \ldots, x_n)$ in the elementary symmetric polynomials $\sigma_1(x_1, \ldots, x_n), \ldots, \sigma_n(x_1, \ldots, x_n)$ (i.e., a polynomial of equivariant Chern classes) is called a **distinguishing polynomial** for M^{T^k} if for any two distinct equivalence classes $[p], [q] \in M^{T^k}/\sim$,

$$g(\lambda_{p,1}, \ldots, \lambda_{p,n}) \neq g(\lambda_{q,1}, \ldots, \lambda_{q,n}) \quad \text{in } H^*(BT^k).$$

Outline of Proof for Main Result

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Proof.

Two points:

- Use of the Fundamental Theorem of Algebra;
- Use of the Uniqueness Theorem for N-adic representations. [a]

□

[a] G. H. Hardy, and E. M. Wright, An Introduction to the Theory of Numbers. 6th edition. Oxford University Press, 2008.

Outline of Proof for Main Result

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- Assume that g is a distinguishing polynomial such that for all $l < |M^{T^k}|$,

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- Write $M^{T^k}/\sim = \{[p_1], ..., [p_m]\}$, let l run over $0, 1, ..., m-1 < |M^{T^k}|$, and then we obtain a system of equations

$$\begin{pmatrix} 1 & \cdots & 1 \\ g(\lambda_{p_1,1}, ..., \lambda_{p_1,n}) & \cdots & g(\lambda_{p_m,1}, ..., \lambda_{p_m,n}) \\ \vdots & \ddots & \vdots \\ g(\lambda_{p_1,1}, ..., \lambda_{p_1,n})^{m-1} & \cdots & g(\lambda_{p_m,1}, ..., \lambda_{p_m,n})^{m-1} \end{pmatrix} \begin{pmatrix} \frac{\varepsilon_{[p_1]}}{\sigma_n(\lambda_{p_1,1}, ..., \lambda_{p_1,n})} \\ \frac{\varepsilon_{[p_2]}}{\sigma_n(\lambda_{p_2,1}, ..., \lambda_{p_2,n})} \\ \vdots \\ \frac{\varepsilon_{[p_m]}}{\sigma_n(\lambda_{p_m,1}, ..., \lambda_{p_m,n})} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Example: top equivariant Chern class is insufficient

- Given integer a and b, let $\phi_{ab} : S^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ be the S^1-action defined by $(s, [z_0 : z_1], [z'_0, z'_1]) \longmapsto ([z_0 : s^a z_1], [z'_0, s^b z'_1])$, fixing four points $p_1 = ([1 : 0], [1 : 0]), p_2 = ([1 : 0], [0 : 1]), p_3 = ([0 : 1], [1 : 0]), p_4 = ([0 : 1], [0 : 1])$ with the corresponding weights $\{a, b\}, \{a, -b\}, \{-a, b\}, \{-a, -b\}$, respectively.

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- Take $(a, b) = (1, 6)$ and $(2, 3)$, a direct calculation gives

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always holds for all $l \geq 1$, but

$$\langle (c_1^{S^1})^2 c_2^{S^1}(TM(1,6)), [M(1,6)]\rangle = 148t^2 \neq 52t^2 = \langle (c_1^{S^1})^2 c_2^{S^1}(TM(2,3)), [M(2,3)]\rangle,$$

implying that $M(1, 6)$ is not equivariantly bordant to $M(2, 3)$.

Non-uniqueness of distinguishing polynomials

In the case $k = n$

Lemma

Let $p, q \in M^{T^n}$. Then the following statements are equivalent.

(1) $c^{T^n}(T_pM) = c^{T^n}(T_qM)$;

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Remark. The above lemma implies that there are two admissible choices for distinguishing polynomials

$$g = c_1^{T^n}(TM) + c_2^{T^n}(TM) \text{ or } g = c_1^{T^n}(TM) + c_n^{T^n}(TM)$$

demonstrating that multiple valid choices of g are available.

Minimal distinguishing degree

For a $2n$-dimensional unitary T^k-manifold M^{2n} with a finite fixed point set M^{T^k}, define the **minimal distinguishing degree**

$$\mathbf{m}(k, M^{2n}) = \min\{\deg g | g \text{ is a distinguishing polynomial}\}.$$

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Corollary (Dimensionally constrained vanishing result)

Suppose that M^{2n} is a $2n$-dimensional unitary T^k-manifold with a finite fixed point set. If $\mathbf{m}(k, M^{2n})\chi(M^{2n}) \leq 2n$, then M^{2n} bounds equivariantly.

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This captures the fundamental interplay between $\chi(M)$ and the dimension of M, mediated by $\mathbf{m}(k, M^{2n})$.

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Corollary

Suppose M^{2n} is a $2n$-dimensional unitary T^k-manifold with a finite fixed point set, satisfying further that M^{2n} bounds non-equivariantly, but does not bound equivariantly. Then

$$\mathbf{m}(k, M^{2n})\chi(M^{2n}) > 2n + \mathbf{m}(k, M^{2n}). \tag{2}$$

Computability of minimal distinguishing degree

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- We conclude that $\mathbf{m}(1, M(a,b)) = 2$ regardless of whether $a \neq b$ holds or not.

Application to Kosniowski's conjecture

Conjecture (Kosniowski)

Suppose that M^{2n} is a unitary S^1-manifold with isolated fixed points. If M^{2n} is not a boundary, then the number of fixed points is greater than a positive linear function $f(n)$ of n.[a]

[a] C. Kosniowski. Some formulae and conjectures associated with circle actions. In *Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979)*, volume 788 of *Lecture Notes in Math.*, pages 331–339. Springer, Berlin, 1980.

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Wen and Ma[1] extended Conjecture to the T^k-equivariant setting, formulating a generalized version recorded as Conjecture 2 therein, which we here refer to as the **toric generalization**.

[1] S. Y. Wen and J. Ma. Number of fixed points for unitary T^{n-1}-manifold. *Front. Math. China*, 14(4):819–831, 2019.

Existence of a positive linear function $f(n)$

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We formalize this result as follows.

Theorem (Chen-L-Yang)

Suppose that M^{2n} is a unitary T^k-manifold with isolated fixed points. If M^{2n} is not a boundary, then

$$\chi(M^{2n}) > \frac{2n}{\boldsymbol{m}(k, M^{2n})}.$$

The lower bound $f(n) = \frac{n}{2}$

Substituting the candidate function $f(n) = \frac{n}{2}$, both Kosniowski's conjecture and its toric generalization reduce to the inequality

$$2\chi(M^{2n}) > n.$$

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[3] P. Li and K. F. Liu. Some remarks on circle action on manifolds. *Math. Res. Lett.*, 18:437–446, 2011.

[4] A. Pelayo and S. Tolman. Fixed points of symplectic periodic flows. *Ergodic Theory and Dynamical Systems*, 31:1237–1247, 2011.

[5] M. Wiemeler. On circle actions with exactly three fixed points. *J. Fixed Point Theory Appl.*, 27(3):Paper No. 79, 12 pp., 2025.

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- When $k = 1$, partial progress supporting this inequality $2\chi(M^{2n}) > n$ has been achieved under restrictions such as $\chi(M^{2n}) \leq 3$ or additional geometric and algebraic constraints for unitary S^1-manifolds.[1] [2] [3] [4] [5]

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- We note that the key lemma employed to attack Kosniowski's conjecture contains a logical gap in the paper.[6]

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The case $n = k$

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We summarize as follows.

Proposition

The toric generalization of Kosniowski's conjecture to the inequality $2\chi(M^{2n}) > n$ holds for $k = n$.

The case $n = k + 1$

- Wen and Ma treated the case $k = n - 1$ and proved the weaker lower bound $\lceil \frac{n}{6} \rceil + 1$ for $\chi(M^{2n})$, which only gives the looser estimate $6\chi(M^{2n}) > n$, where $\lceil \cdot \rceil$ denotes the ceiling function.

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- We examine the unitary T^2-manifold $S^6 \cong G_2/SU(3)$ and verify that the value $\boldsymbol{m}(2, S^6) = 6$. Substituting this into our main inequality $\boldsymbol{m}(2, S^6)\chi(S^6) > 6$ simplifies to the sharp bound $\chi(S^6) > 1$, consistent with the estimate $2\chi(M^{2n}) > n$.

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- This example shows that the inequality $\mathbf{m}(k, M^{2n})\chi(M) > 2n$ may lead to a sharp inequality even for $\mathbf{m}(k, M^{2n}) > 4$, from which we can further induce $2\chi(M^{2n}) > n$.

Calculation of $\mathbf{m}(2, S^6) = 6$

The canonical action of a common maximal torus T^2 on $S^6 = G_2/SU(3)$ has exactly two isolated fixed points $p_\pm$. The weight sets of the tangent representations at two fixed points p_+ and p_- are

$$W_+ = \{(1,0),\ (-1,1),\ (0,-1)\},\quad W_- = \{(-1,0),\ (1,-1),\ (0,1)\}$$

So the total equivariant Chern classes of the tangent representations $T_{p_+}S^6$ and $T_{p_-}S^6$ are

$$c^{T^2}(T_{p_+}S^6) = (1+t_1)(1-t_1+t_2)(1-t_2),\quad c^{T^2}(T_{p_-}S^6) = (1-t_1)(1+t_1-t_2)(1+t_2)$$

respectively, in $H^*(BT^2) = \mathbb{Z}[t_1, t_2]$. Furthermore, it follows that

$$c_1^{T^2}(T_{p_+}S^6) = c_1^{T^2}(T_{p_-}S^6) = 0,\quad c_2^{T^2}(T_{p_+}S^6) = c_2^{T^2}(T_{p_-}S^6) = t_1t_2 - t_1^2 - t_2^2,$$

and $c_3^{T^2}(T_{p_+}S^6) = t_1t_2(t_1-t_2) \neq -t_1t_2(t_1-t_2) = c_3^{T^2}(T_{p_-}S^6)$, thus concluding that $\mathbf{m}(2, S^6) = 6$.

The lower bound sharper than $f(n) = \frac{n}{2}$

In what follows, we can see that $f(n) = \dfrac{n}{2}$ may not be a sharp lower bound.

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Recall

Corollary

Suppose M^{2n} is a $2n$-dimensional unitary T^k-manifold with a finite fixed point set, satisfying further that M^{2n} bounds non-equivariantly, but does not bound equivariantly. Then

$$\mathbf{m}(k, M^{2n})\chi(M^{2n}) > 2n + \mathbf{m}(k, M^{2n}). \tag{3}$$

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By rearranging the inequality (3)

$$\chi(M^{2n}) > \frac{2n}{m(k, M^{2n})} + 1,$$

which is significantly stronger than the candidate function $f(n) = \frac{n}{2}$ noted by Kosniowski, as shown below:

$$\chi(M^{2n}) > \frac{2n}{\mathbf{m}(k, M^{2n})} + 1 \geq \frac{n}{2} + 1 > \frac{n}{2} = f(n)$$

whenever $\mathbf{m}(k, M^{2n}) \leq 4$.

An observation

- It is well-known [7] that any closed connected smooth orientable (or unitary) manifold N admitting an effective T^k-action with $k \geq 1$ contains a circle subgroup $S \subset T^k$ satisfying $N^S = N^{T^k}$.

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- Consequently, the validity of $2\chi(M^{2n}) > n$ for all unitary S^1-manifolds automatically implies the same bound for arbitrary unitary T^k-manifolds with $k > 1$.

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- The possibility of counterexamples

$$\mathbf{m}(1, M^{2n})\chi(M^{2n}) > 2n \geq 4\chi(M^{2n}).$$

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Thank You!