Topological obstructions to geometric positivity on Calabi-Yau manifolds

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Example

The complete intersections

$$X_n(d_1, \ldots, d_r) \subset \mathbb{P}^{n+r}, \quad c_1 = n + r + 1 - \sum_{i=1}^{r} d_i,$$

which are Calabi-Yau, Fano, or canonically polarized if $n + r + 1 =$, $>$ or $< \sum_{i=1}^{r} d_i$.

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  - $M_- \ncong_{\text{top}} M_0$: by the Hirzebruch signature

$$3\sigma = c_1^2 - 2c_2$$

and the Miyaoka-Yau

$$c_1^2 \leq 3c_2.$$

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Question (Nakamura, Campana-Peternell 1990s)

whether there exists a Calabi-Yau threefold that is (orientation-preservingly) homeomorphic to either $X_3(3) \subset \mathbb{P}^4$, or $X_3(2,2) \subset \mathbb{P}^5$?

Question (widely open?)

Whether $M_+ \not\cong_{top} M_0$ *and* $M_- \not\cong_{top} M_0$*?*

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Theorem (Oguiso-Peternell 2020)

Let M_0 *be even dimension* $n \geq 4$*. If* $n \geq 6$*, assume in addition that* $b_2(M_0) = 1$*. Then* M_0 *is not orientation-preservingly homeomorphic to an* M_+*.*

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Let M_0 be even dimension $n \geq 4$. If $n \geq 6$, assume in addition that $b_2(M_0) = 1$. Then M_0 is not orientation-preservingly homeomorphic to an M_+.

Proof strategies: assume on the contrary that $M_0 \cong M_+$.

1. Show that $\mathrm{Todd}(M_0) = 0$.
2. Show that $M_0 = M_1 \times M_2$, where M_1 is an odd-dimensional irreducible Calabi-Yau manifold, and M_2 is either a point or a product of some irreducible Calabi-Yau and hyperKähler manifolds.

- Main tools: H-R-R ideas (following Kodaira-Hirzebruch), the Wu formula, the Bogomolov-Beauville decomposition.

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3. Show that

$$\hat{A}(M) = \chi\big(M, \mathcal{O}_M(-L)\big) = 0.$$

Corollary (L. 2026)

Let M_0 be even dimension $n \geq 4$. If $n \geq 6$, assume in addition that $b_2(M_0) = 1$. Then M_0 cannot be homeomorphic to a Kähler manifold M with $c_1(M)$ quasi-positive.

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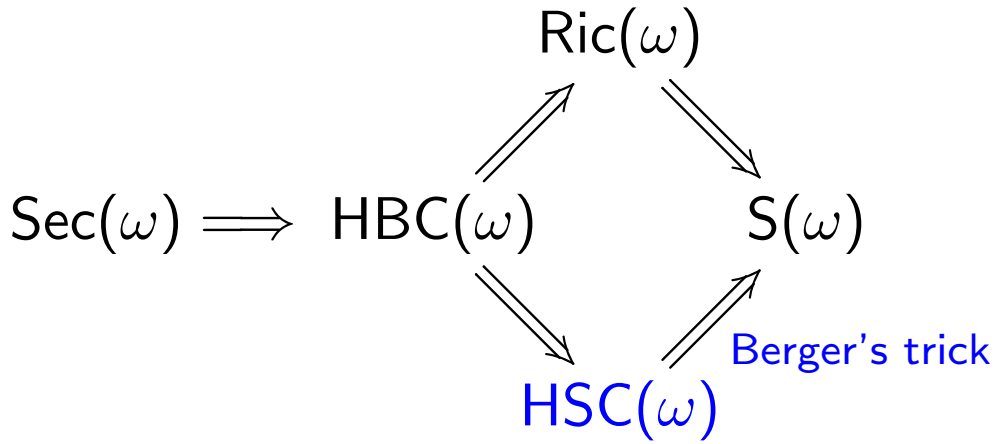
- (M, g): Riemannian manifold ("$\Longrightarrow$": domination)

$$\mathrm{Sec}(g) \Longrightarrow \mathrm{Ric}(g) \Longrightarrow \mathrm{S}(g).$$

$$\begin{cases} \mathrm{Sec}(u \wedge v) := R(u, v, v, u) \quad \text{(sectional curvature)} \\ \mathrm{Ric}(u) := \sum_i \mathrm{Sec}(u \wedge e_i) \quad \text{(Ricci curvature)} \\ \mathrm{S}(g) := \sum_i \mathrm{Ric}(e_i) \quad \text{(scalar curvature)} \end{cases}$$

- (M, ω): Kähler manifold



$$\begin{cases} \mathrm{HBC}(u,v) := R(u, Ju, Jv, v) = \mathrm{Sec}(u \wedge Ju) + \mathrm{Sec}(v \wedge Jv) \\ \text{(holomorphic bisectional curvature)} \\ \\ \mathrm{HSC}(u) := R(u, Ju, Ju, u) \\ \text{(holomorphic sectional curvature)} \end{cases}$$

(Berger's trick)
Choose a *unitary* basis $\{e_1, \ldots, e_n\}$ of $T_p^{1,0}M$.

$$\Sigma_p := \left\{X = \sum_{i=1}^{n} X^i e_i \in T_p^{1,0}M,\ |X| = 1\right\} \cong \mathbb{S}^{2n-1}.$$

$$\int_{X\in\Sigma_p} \mathrm{HSC}_p(X) d\theta(X) \quad (d\theta(\cdot)\text{: spherical measure})$$

$$= \int_{X\in\Sigma_p} \left[R(e_i, \overline{e_j}, e_k, \overline{e_l}) X^i \overline{X^j} X^k \overline{X^l}\right] d\theta(X)$$

$$= R(e_i, \overline{e_j}, e_k, \overline{e_l}) \frac{\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}}{n(n+1)} \mathrm{Vol}(\mathbb{S}^{2n-1})$$

$$= \frac{\mathrm{Vol}(\mathbb{S}^{2n-1})}{n(n+1)} S_g(p).$$

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Let M_0 be even dimension $n \geq 4$. If $n \geq 6$, assume in addition that $b_2(M_0) = 1$. Then M_0 is not homeomorphic to a Kähler (M, ω) with HSC(ω) quasi-positive.

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Proof strategies: assume that $M_0 \cong_{\text{top}} M$.

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Corollary

No M_0 is homeomorphic to a symplectic manifold admitting a Hamiltonian circle action with isolated fixed points.

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Proof strategies. Assume that $M \cong_{\text{top}} M_0$.

1. $$\hat{A}(M) = \hat{A}(M_0) = 0.$$

2. $$M_0 = M_1 \times M_2,$$

   where M_1 an odd-dim irreducible CY manifold and M_2 either a point or a product of irreducible CY and hyperKähler factors. This implies that some odd Betti number of M_0 is nonzero.

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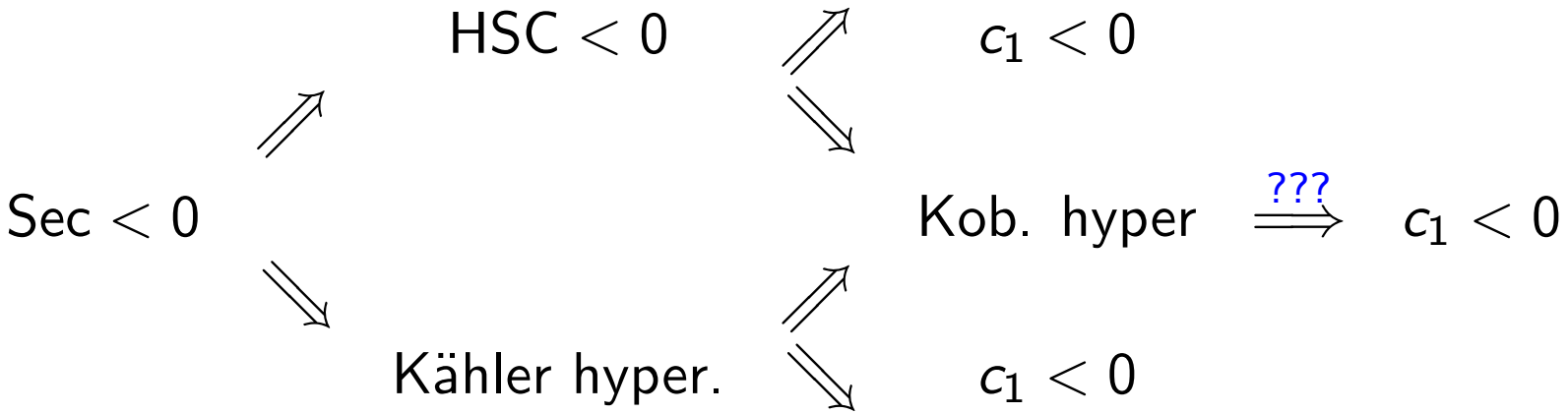

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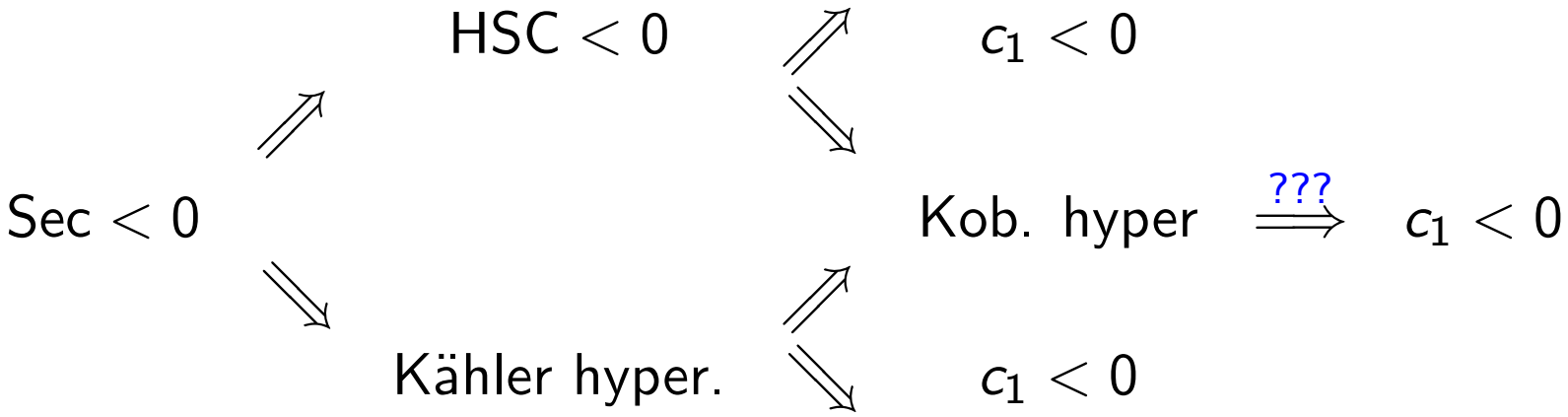

Conjecture (Kobayashi)

M_0 is not Kobayashi hyperbolic, More generally, Kobayashi hyperbolicity implies $c_1 < 0$.

Proposition (L.)

Let (M, g) be a Kähler hyperbolic manifold. Then $\pi_1(M_0)$ is not isomorphic to $\pi_1(M)$. Consequently, M_0 and M are not homotopy equivalent.

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Proof strategies: assume that $\pi_1(M_0) \cong \pi_1(M)$.

1. $\pi_1(M)$ is amenable and infinite, which implies that the Euler characteristic $\chi(M) = 0$ (Cheeger-Gromov 1986).
2. Kähler hyperbolicity implies that $(-1)^n \chi(M) > 0$ (Gromov 1991).

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Fact

No Calabi-Yau surface admits a Riemannian metric of positive scalar curvature.

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Fact

No Calabi-Yau surface admits a Riemannian metric of positive scalar curvature.

A finite cover of it is either a complex torus T or a $K3$ surface.

When $n \geq 3$, we have

Example

Let E be an elliptic curve and $X(5)$ a smooth quintic threefold in $\mathbb{P}^4$. Set

$$M_1 := \underbrace{E \times \cdots \times E}_{n \text{ copies}}, \quad M_2 := \begin{cases} X(5), & n = 3, \\ X(5) \times \underbrace{E \times \cdots \times E}_{n-3 \text{ copies}}, & n \geq 4, \end{cases}$$

Both M_1 and M_2 are n-dimensional Calabi-Yau manifolds. The first admits no PSC metric, whereas the second does.

谢谢!