On the boundedness of almost flat spin^c manifolds

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Section 1

Almost flat and earlier work

Flat manifolds bound

Flat The Riemann curvature tensor vanishes everywhere.

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Theorem (Hamrick–Royster, 1982)

Every closed flat manifold bounds a compact smooth manifold.

Almost flat manifolds

Definition For every $\varepsilon > 0$, M admits g_ε with

$$\operatorname{diam}(M, g_\varepsilon) \leq 1, \qquad |K_{g_\varepsilon}| < \varepsilon.$$

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Example The Heisenberg nilmanifold

$$S^1 \longrightarrow H_3(\mathbb{Z})\backslash H_3(\mathbb{R}) \longrightarrow T^2$$

is almost flat and nonflat.

The Farrell–Zdravkovska–Yau conjecture

Theorem (Gromov, 1978, refined by Ruh, 1982)
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Conjecture (Farrell–Zdravkovska, 1983 and Yau, 1993)
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Known cases

- Thuong (2015) Dimension four.

- Davis–Fang (2016), Xiong (2016)
  Cyclic or generalized quaternion Sylow 2-subgroups.

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- Thuong (2016)
  Two-generator abelian or dihedral holonomy,
  with additional conditions on the action on the center.

Section 2

Null-bordism of almost flat spin^c manifolds

Low-dimensional null-bordism

Theorem (Han–Huang–Zhang, 2025)
Let M^n be almost flat and spin^c, with $n \leq 9$. Then

$$[M] = 0 \quad \text{in } \Omega_n^{SO}.$$

Low-dimensional null-bordism

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Let M^n be almost flat and spin^c, with $n \leq 9$. Then

$$[M] = 0 \quad \text{in } \Omega_n^{SO}.$$

If n is odd, then for every spin^c structure c,

$$[M, c] = 0 \quad \text{in } \Omega_n^{\text{spin}^c}.$$

Spin^c manifolds

For oriented M,

$$M \text{ is spin}^c \iff \omega_2(M) = \rho_2(c) \text{ for some } c \in H^2(M;\mathbb{Z}).$$

The determinant line ξ has $c_1(\xi) = c$.

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Examples $h = c_1(\mathcal{O}(1))$.

M	$\omega_2(M)$	c (canonical)
$\mathbb{CP}^1$	0	$2h$
$\mathbb{CP}^2$	$\rho_2(h) \neq 0$	$3h$

Our methods

Earlier approach Group theoretic arguments.

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Our approach

Integral Wu classes + spin^c Atiyah–Singer index theorem

$$\underbrace{\mathrm{Ind}(D_c) \in \mathbb{Z}}_{\text{index theorem}} \implies \underbrace{\text{integral Wu numbers are even}}_{\dim M = 2m > 0}.$$

Bordism criteria

Theorem (Wall, 1960)

For a closed oriented manifold M,

$$[M] = 0 \text{ in } \Omega_n^{SO} \iff \begin{array}{c}\text{Pontryagin numbers} = 0,\\ \text{Stiefel–Whitney numbers} = 0.\end{array}$$

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Theorem (Stong, 1968)
For a closed spin^c manifold (M, c),

$$[M, c] = 0 \text{ in } \Omega_n^{\text{spin}^c} \iff \begin{array}{c}c\text{-Pontryagin numbers} = 0,\\ \text{Stiefel–Whitney numbers} = 0.\end{array}$$

Section 3

Integral Wu classes and the index theorem

Tangential Wu classes

For $x \in H^{n-i}(M; \mathbb{Z}/2)$,

$$\langle \mathrm{Sq}^i x, [M] \bmod 2 \rangle = \langle \mu_i x, [M] \bmod 2 \rangle.$$

$$\mu_t = 1 + \mu_1 + \mu_2 + \cdots, \qquad \omega_t = \mathrm{Sq}(\mu_t).$$

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Characteristic series

$$h(x) = 1 + x + x^2 + x^4 + x^8 + \cdots = 1 + \sum_{k \geq 0} x^{2^k}.$$

Integral Wu classes

We define a tangential spin Wu class

$$\mu_t^{\mathrm{Spin}}(V) = 1 + \mu_4^{\mathrm{Spin}}(V) + \mu_8^{\mathrm{Spin}}(V) + \cdots \in H^{4*}(M;\mathbb{Z}),$$

with rational characteristic series

$$G(x) = \sqrt{h(x)h(-x)} = 1 + \frac{x^2}{2} + \frac{11x^4}{8} + \frac{5x^6}{16} + \cdots .$$

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For a spin^c bundle (V,ξ), $c = c_1(\xi)$, we define

$$\mu_t^{\mathrm{Spin}^c}(V) := \frac{\mu_t^{\mathrm{Spin}}(V\oplus\xi)}{\mu_t^{\mathrm{U}}(\xi)} = \frac{\mu_t^{\mathrm{Spin}}(V\oplus\xi)}{h(c)}, \qquad \rho_2(\mu_t^{\mathrm{Spin}^c}) = \mu_t.$$

Integral spin^c Wu classes

Example In rational cohomology,

$$\mu_2^{\mathrm{Spin}^c} = -c,$$

$$\mu_4^{\mathrm{Spin}^c} = \frac{c^2}{2} + \frac{p_1}{2},$$

$$\mu_6^{\mathrm{Spin}^c} = \frac{c^3}{2} - \frac{cp_1}{2},$$

$$\mu_8^{\mathrm{Spin}^c} = -\frac{5c^4}{8} + \frac{c^2p_1}{4} + \frac{11p_1^2}{8} - \frac{5p_2}{2},$$

$$\mu_{10}^{\mathrm{Spin}^c} = \frac{9c^5}{8} + \frac{c^3p_1}{4} - \frac{11cp_1^2}{8} + \frac{5cp_2}{2}.$$

The index and divisibility

Lemma

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For an almost flat spin^c manifold (M^{2m}, c),

$$\mathrm{Ind}(D_c) = \langle \widehat{A}(M) e^{c/2}, [M] \rangle \in \mathbb{Z}.$$

Lemma (Han–Huang–Zhang, 2025)

$$\langle c^m, [M] \rangle = 2^m m! \, \mathrm{Ind}(D_c).$$

Wu numbers

Proposition (Han–Huang–Zhang, 2025)
For an almost flat spin^c manifold M^{2m} and a partition I of m,

$$2^m \mid \langle \mu_{2I}^{\mathrm{Spin}^c}(M), [M] \rangle.$$

Wu numbers

Proposition (Han–Huang–Zhang, 2025)
For an almost flat spin^c manifold M^{2m} and a partition I of m,

$$2^m \mid \langle \mu_{2I}^{\mathrm{Spin}^c}(M), [M] \rangle.$$

Corollary (Han–Huang–Zhang, 2025)
All mod-two Wu numbers of an almost flat spin^c manifold vanish.

Dimensions at most nine

Lemma (Han–Huang–Zhang, 2025)

For a closed spin^c manifold M^n, $n \leq 9$, every top-degree Stiefel–Whitney monomial is a polynomial in the Wu classes.

$$\omega_I(M) = P_I\Big(\mu_2(M), \mu_4(M), \dots\Big).$$

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$$\omega_I(M) = P_I\Big(\mu_2(M), \mu_4(M), \dots\Big).$$

For almost flat M, all these characteristic numbers vanish. Hence

$$[M] = 0 \text{ in } \Omega_n^{SO}.$$

Dimension ten

There exists a closed spin^c 10-manifold N with

$$p_i(N) = 0 \text{ in } H^{4i}(N;\mathbb{Q}) \quad (i > 0),$$

all mod-two Wu numbers zero,

$$\langle \omega_4(N)\omega_6(N), [N] \bmod 2 \rangle = 1.$$

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If $\mathrm{Sq}^2\omega_4 = 0$ on a spin^c manifold, then

$$\omega_4\omega_6 = \omega_2\omega_4^2 = \mu_2(\mu_4 + \mu_2^2)^2.$$

Higher dimensions

$$U(M) = (\mathbb{Z}/2)[\mu_2(M), \mu_4(M), \ldots] \subset H^*(M; \mathbb{Z}/2).$$

Theorem (Han–Huang–Zhang, 2026)

Let M^n be almost flat and spin^c, with $n \leq 9 \cdot 2^\ell$, $\ell \in \mathbb{Z}_{\geq 0}$. If $\omega_{3 \cdot 2^a}(M) \in U(M)$ for $1 \leq a \leq \ell$, then $[M] = 0$ in Ω_n^{SO}.

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Sufficient conditions

$$\begin{aligned}
n \leq 18 \quad & \mathrm{Sq}^2 \omega_4 = 0 \\
n \leq 36 \quad & \mathrm{Sq}^2 \omega_4 = 0, \quad \mathrm{Sq}^4 \omega_8 = 0 \\
n \leq 72 \quad & \mathrm{Sq}^2 \omega_4 = 0, \quad \mathrm{Sq}^4 \omega_8 = 0, \quad \mathrm{Sq}^8 \omega_{16} = 0
\end{aligned}$$

Section 4

Holonomy and characteristic classes

The holonomy group

(Gromov, 1978, refined by Ruh, 1982)

$$M \cong \Gamma \backslash (L \rtimes G)/G.$$

- L is a simply connected nilpotent Lie group.
- $\Gamma \leq L \rtimes G$ is discrete, torsion-free and cocompact.
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$$1 \longrightarrow \Gamma \cap L \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

The finite group G is the holonomy group of M.

Holonomy and null-bordism

Theorem (Han–Huang–Zhang, 2026)

Let M be almost flat and spin^c. It bounds orientably if a Sylow 2-subgroup P of its holonomy group is

$$\begin{aligned} &\mathbb{Z}/2^a \oplus \mathbb{Z}/2^b && a \geq b \geq 0, \\ &D_{2^s},\ Q_{2^s} && s \geq 3, \\ &SD_{2^s},\ M_{2^s} && s \geq 4. \end{aligned}$$

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Davis–Fang (2016) Cyclic or generalized quaternion.

Additional spin^c cases Rank-two abelian, dihedral, semidihedral and modular groups.

A null-bordism criterion

Proposition (Han–Huang–Zhang, 2026)

Let M be almost flat and spin^c. Suppose

$$\omega_t(TM) = \omega_t(\eta)\rho_2(c(E)),$$

where

- η is an oriented real bundle of rank at most three,
- E is a complex bundle with finite holonomy.

Then M bounds an orientable manifold.

Geometric consequence

Theorem (Ontaneda, 2020)

If an almost flat M bounds, it bounds a compact W whose interior has a complete, finite-volume metric of pinched negative sectional curvature.

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Our cases Almost flat spin^c manifolds with

- dimension $n \leq 9$, or
- one of the listed Sylow 2-subgroups.

Selected references

F. Han, R. Huang, W. Zhang
On the boundedness of almost flat spin^c *manifolds.* October 2026.

F. T. Farrell, S. Zdravkovska
Do almost flat manifolds bound? 1983.

M. J. Hopkins, I. M. Singer
Quadratic functions in geometry, topology, and M-theory. 2005.

J. F. Davis, F. Fang
An almost flat manifold with a cyclic or quaternionic holonomy group bounds. 2016.

Thank you!