Topological complexity of enumerative problems in algebraic geometry

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Overview

1. Backgrounds
2. The topological complexity of enumerative problems
3. Finding lower bounds by pullbacks
4. The cohomology of PU_4/K

Backgrounds

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Theorem (the fundamental theorem of algebra)

Every non-constant single-variable polynomial with complex coefficients of degree d has d complex roots, counted with multiplicity.

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In 1987, S. Smale [5] considered the topological complexity of finding the d roots of a generic degree-d polynomial:

Poly(d)

Given a generic (i.e., with no repeated roots) degree d polynomial in one variable and $\epsilon > 0$, find all roots of f within ϵ.

Theorem (Smale, [5], 1987)

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Roughly speaking, the topological complexity of an algorithm measures the number of "branches" in the algorithm.

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Roughly speaking, the topological complexity of an algorithm measures the number of "branches" in the algorithm. The lower bound of Smale is later improved by Vassiliev, De Concini-Procesi-Salvetti, and Arone.

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Examples of enumerations in algebraic geometry (over $\mathbb{C}$):

1. a generic degree d-polynomial has d roots;
2. a smooth cubic surface contains 27 lines;
3. a smooth quartic plane curve has 28 bitangent lines;
4. a generic quartic plane curve has 24 flex points;
5. a smooth cubic plane curve has 9 flex points;
6. ...

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A positive integer, which we call the *topological complexity* may be assigned to each of the enumerative problems in algebraic geometry.

Broader context

A general question

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  complexity $\sim$ resolvent degrees (Brauer 1975)
- find = approximate using an algorithm
  complexity $\sim$ topological complexity (Smale 1987)

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- Galois groups of enumerative problems (Jordan, 1870, Harris 1979)
- resolvent degrees of enumerative problems (Farb-Wolfson 2019)
- topological complexity of enumerative problems (this talk).

The topological complexity of enumerative problems

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Following Smale, by an *algorithm* we mean a finite rooted tree consisting of a root for the input, leaves for the output, and internal nodes of the following two types:

computation nodes  *and* *branching nodes*  .

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The topological complexity of a problem P is the minimum of the topological complexity of any algorithm solving P.
Smale [5] first proved a lower bound for the topological complexity of the problem of finding roots of a polynomial $f(z) = 0$. Smale's lower bounds were later improved by Vassiliev [6], De Concini-Procesi-Salvetti [3], and Arone [1].

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Similar questions may be asked about solutions to enumerative problems for high-dimensional objects such as curves and surfaces, which is mostly unknown. A first step in this direction is the work [2] by Weiyan Chen and Zheyan Wan, concerning the topological complexity of finding inflection points on cubic plane curves. In this talk, we consider **the lower bounds of the topological complexity** of the following problems:

- **Line**(ϵ): Given any cubic surface defined by a homogeneous polynomial $F(x, y, z, w)$ of degree 3, find all of its 27 lines $(l_1, \cdots, l_{27})$ within ϵ.
- **Bitangent**(ϵ): Given any quartic curve defined by a homogeneous polynomial $F(x, y, z)$ of degree 4, find all of its 28 bitangent lines $(l_1, \cdots, l_{28})$ within ϵ.
- **Flex4**(ϵ): Given any quartic curve defined by a homogeneous polynomial $F(x, y, z)$ of degree 4 with 24 distinct flex points, find all of its 24 inflection points $(p_1, \cdots, p_{24})$ within ϵ.

The topological complexity of enumerative problems

Theorem (Chen-G. 2024)

When ϵ is sufficiently small, we have

1. *the topological complexity of the problem* **Line**(ϵ) *is at least* 15,
2. *the topological complexity of the problem* **Bitangent**(ϵ) *is at least* 8,
3. *the topological complexity of the problem* **Flex4**(ϵ) *is at least* 8.

The topological complexity of enumerative problems

Another problem under consideration is the following:

Flex3(ϵ): Given any smooth cubic curve defined by a homogeneous polynomial $F(x, y, z)$ of degree 3 with 9 distinct flex points, find all of its 9 flex points $(p_1, \cdots, p_9)$ within ϵ.

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When ϵ is sufficiently small, the topological complexity of the problem **Flex3**(ϵ) *is at least 7.*

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Theorem (Chen-G. AI assisted, 2026)

When ϵ is sufficiently small, the topological complexity of the problem **Flex3**(ϵ) *is 8.*

The topological construction associated to the enumerative problems

General idea

- An enumerative problem P has an associated "space of problems" B_P. Each point of B_P corresponds to a specific problems with d solutions.

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- The map $E_P \to B_P$ given by $(b, s) \mapsto b$ is a d sheeted covering map.
- The topological complexity of P roughly corresponds to the minimum number of open sets in a trivializing cover of B_P with respect to the covering map $E_P \to B_P$.

The topological construction associated to the enumerative problems

There are various parameter spaces associated to the three enumerative problems. For instance, for the problem **Line**(ϵ), consider

$B_{\text{line}} := \{\text{nonsingular homogeneous cubic polynomials } F(x,y,z,w)\}/\mathbb{C}^\times$

$E_{\text{line}} := \{(F, l_1, \cdots, l_{27}) : \ l_i\text{'s are the 27 lines on the cubic surface } F \in B_{\text{line}}\}$

The space B_{line} is an open submanifold of $\mathbb{C}P^{19}$ consisting of all homogeneous cubic polynomials in four variables.

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The projection $E_{\text{line}} \to B_{\text{line}}$ given by $(F, l_1, \cdots, l_{27}) \mapsto F$ is a S_{27}-cover, where S_{27} acts on E_{line} by permuting the ordering of the 27 lines.

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Similarly, we have a S_{28}-cover $E_{\text{btg}} \to B_{\text{btg}}$ associated to the problem **Bitangent**(ϵ) and a S_{24}-cover $E_{\text{flex}} \to B_{\text{flex}}$ associated to the problem **Flex4**(ϵ).

The topological construction associated to the enumerative problems

Definition (Schwarz, [4])

The *Schwarz genus* of a covering $E \to B$, denoted by $g(E \to B)$ or simply $g(E)$, is the minimum size of an open cover of B consisting of open sets such that there exists a continuous section of the covering map $E \to B$ over each open set.

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Theorem (Schwarz, [4])

The topological complexity of an enumerative problem P is $g(E_P \to B_P) - 1$.

The topological construction associated to the enumerative problems

Proposition (Schwarz, [4], p.71)

Let $i^*E \to B'$ denote the pullback of a cover $E \to B$ along a continuous map $i : B' \to B$. Then $g(i^*E \to B') \leq g(E \to B)$.

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If B is a CW complex of dimension d, then $g(E \to B) \leq d + 1$.

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Proposition (Disconnected covers)

Consider a cover $E \to B$ with path-components $E = \bigcup_{i \in I} E_i$ where each $E_i \to B$ is also a cover. Suppose that there exists an $m \in I$ such that for any $i \in I$, there exists morphism of coverings $E_i \to E_m$. Then $g(E) = g(E_m)$.

The topological construction associated to the enumerative problems

Proposition (Schwarz, [4], p.98)

Suppose that $E \to B$ is a a principal Γ-bundle with a classifying map $cl : B \to B\Gamma$ where $B\Gamma$ is the classifying space of Γ. Then we have

$$g(E \to B) \geq \min\{k : \mathrm{H}^i(B\Gamma; A) \xrightarrow{cl^*} \mathrm{H}^i(B; A) \text{ is zero for any } i \geq k\}$$

for any Γ-module A. The integer on the right hand side of the inequality above is called the *homological A-genus* of the cover $E \to B$ and is denoted by $g_A(E \to B)$.

The topological construction associated to the enumerative problems

Summary

Let $TC(P)$ be the topological complexity of an enumerative problem P. Let $E_P \to B_P$ be the covering space associated to the problem P. Let $E \to B$ be a pullback of $E_P \to B_P$. Let $g(-)$ and $g_A(-)$ be the Schwarz genus and A-genus, respectively. Then we have

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$$TC(P) = g(E_P \to B_P) - 1 \geq g(E \to B) - 1 \geq g_A(E \to B) - 1.$$

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$$TC(P) = g(E_P \to B_P) - 1 \geq g(E \to B) - 1 \geq g_A(E \to B) - 1.$$

Next, we look for a pullback $E \to B$ such that we can calculate $g_A(E \to B)$.

Finding lower bounds by pullbacks

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The space B_{line} is an open submanifold of $\mathbb{C}P^{19}$ consisting of all homogeneous cubic polynomials in four variables.The projective unitary group $\mathrm{PU}_4 := \mathrm{U}_4/\{\lambda \mathrm{I}_4\}$ acts on B_{line} by acting on the variables (x, y, z, w).

Finding lower bounds by pullbacks

We consider the Fermat cubic surface

$$F(x,y,z,w) = x^3 + y^3 + z^3 + w^3.$$

Consider the following subgroup $K \leq \mathrm{PU}_4$ preserving the Fermat cubic surface F:

$$K := \left\langle \begin{bmatrix} e^{2\pi i/3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle$$
$$\cong (\mathbb{Z}/3\mathbb{Z})^3, \tag{1}$$

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which leaves $F(x,y,z,w)$ fixed.

Finding lower bounds by pullbacks

Lemma

The action of K on the set of 27 lines on the Fermat cubic surface is faithful. In other words, the induced group homomorphism $K \to S_{27}$ is injective.

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Therefore, we have an injective map

$$\eta\colon PU_4/K \to B_{\text{line}},\ [\sigma] \mapsto \sigma F.$$

Finding lower bounds by pullbacks

Proposition (Chen-G., 2024)

The total space of the pullback of η is homeomorphic to a disjoint union of copies of PU_4:

$$\eta^* E_{\mathsf{line}} \cong \bigcup_{S_{27}/K} \mathrm{PU}_4$$

where the components are in bijection with K-cosets in S_{27}. In particular, we have

$$g(\eta^* E_{\mathsf{line}} \to \mathrm{PU}_4/K) = g(\mathrm{PU}_4 \xrightarrow{\phi} PU_4/K),$$

where ϕ is the quotient map, and a principal K-bundle.

Finding lower bounds by pullbacks

Theorem (Chen-G, 2024)

The homomorphism

$$H^*(BK;\mathbb{F}_3) \to H^*(PU_4/K;\mathbb{F}_3)$$

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Corollary

$g(E_{line} \to B_{line}) \geq 16$.

Theorem (Chen-G., 2024)

The topological complexity of the problem **Line**(ϵ) *is no less than* 15.

The arguments for the problems **Bitangent**(ϵ) and **Flex4**(ϵ) are similar, and we have

Theorem (Chen-G., 2024)

The topological complexity of the problems **Line**(ϵ), **Bitangent**(ϵ) *and* **Flex4**(ϵ) *no less than* 15, 8, *and* 8, *respectively.*

The cohomology of PU_4/K

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We have a short exact sequence of Lie groups

$$1 \to K \to \mathrm{PU}_4 \to \mathrm{PU}_4/K \to 1,$$

which induces a homotopy fiber sequence

$$\mathrm{PU}_4/K \xrightarrow{cl} BK \xrightarrow{\varphi} B\mathrm{PU}_4$$

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where cl is the classifying map of the principal K-bundle $\mathrm{PU}_4 \to \mathrm{PU}_4/K$,and φ is the map induced by the inclusion $K \to \mathrm{PU}_4$.

The cohomology of PU_4/K

Consider the map $BK \xrightarrow{\varphi} B\mathrm{PU}_4$. We have

$$H^*(B\mathrm{PU}_4; \mathbb{F}_3) \cong \mathbb{F}_3[\epsilon_4, \epsilon_6, \epsilon_8], \ \deg \epsilon_i = i,$$

and

$$H^*(BK; \mathbb{F}_3) \cong \mathbb{F}_3[\xi_1, \xi_2, \xi_3] \otimes \Lambda_{\mathbb{F}_3}[u_1, u_2, u_3],$$

where $\deg \xi_i = 2$, $\deg u_i = 1$.

$$\begin{aligned}
\varphi^*(\epsilon_4) &= 3\sigma_1^2 - 8\sigma_2, \\
\varphi^*(\epsilon_6) &= \sigma_1^3 - 4\sigma_1\sigma_2 + 8\sigma_3, \\
\varphi^*(\epsilon_8) &= 3\sigma_1^4 - 16\sigma_1^2 c_2 + 64\sigma_1\sigma_3 - 256\sigma_4,
\end{aligned}$$

where $\sigma_1 = \xi_1 + \xi_2 + \xi_3$, $\sigma_2 = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1$, and $\sigma_3 = \xi_1\xi_2\xi_3$.

The cohomology of PU_4/K

The homomorphism $\varphi^*\colon H^*(B\mathrm{PU}_4;\mathbb{F}_3) \to H^*(BK;\mathbb{F}_3)$ makes $H^*(BK;\mathbb{F}_3)$ into a $H^*(B\mathrm{PU}_4;\mathbb{F}_3)$-module.

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The Eilenberg-Moore spectral sequence for $\mathrm{PU}_4/K \xrightarrow{cl} BK \xrightarrow{\varphi} B\mathrm{PU}_4$ is of the from

$$E_2 = \mathrm{Tor}_{H^*(B\mathrm{PU}_4;\mathbb{F}_3)}(\mathbb{F}_3, H^*(BK;\mathbb{F}_3)) \Rightarrow H^*(PU_4/K;\mathbb{F}_3).$$

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Using the Koszul resolution of $\mathbb{F}_3$, we compute the E_2-page, concentrated in homological degree 0.

The cohomology of PU_4/K

The homomorphism $\varphi^*\colon H^*(B\mathrm{PU}_4;\mathbb{F}_3) \to H^*(BK;\mathbb{F}_3)$ makes $H^*(BK;\mathbb{F}_3)$ into a $H^*(B\mathrm{PU}_4;\mathbb{F}_3)$-module.

The Eilenberg-Moore spectral sequence for $\mathrm{PU}_4/K \xrightarrow{cl} BK \xrightarrow{\varphi} B\mathrm{PU}_4$ is of the from

$$E_2 = \mathrm{Tor}_{H^*(B\mathrm{PU}_4;\mathbb{F}_3)}(\mathbb{F}_3, H^*(BK;\mathbb{F}_3)) \Rightarrow H^*(PU_4/K;\mathbb{F}_3).$$

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The lower bounds of the topological complexity of **Bitangent**(ϵ) and **Flex4**(ϵ) are obtained in similar ways.

Further Questions

- Better approximations of the lower bounds of the aforementioned problems. This may involve understanding the cohomology of B_{line};

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- geometric interpretations of the cohomology classes of $B\mathrm{PU}(n)$.

Thank You!


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