A Counterexample to Cohomological Rigidity of Toric Manifolds

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Table of contents

1. Background
2. A counterexample
3. More

Toric topology

- Algebraic geometry: Torus embedding [Demazure'70]:

$$\overline{\mathrm{Img}\left((\mathbb{C}^*)^n \hookrightarrow \mathbb{CP}^N\right)}$$

- Combinatorial Geometry: Polytope $P \rightsquigarrow$ toric variety M_P [Stanley'80]:

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor n/2 \rfloor}$$

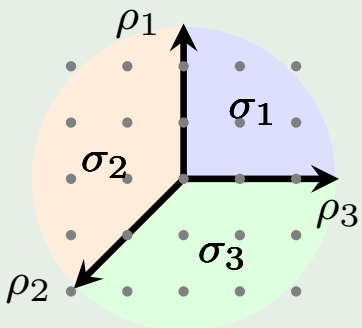
- Symplectic Geometry: Atiyah–Guillemin–Sternberg convexity theorem: Hamiltonian $T \curvearrowright (M, \omega)$, the image of the moment map is convex.

Toric varieties

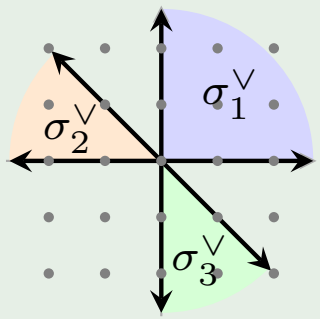
A **toric variety** is a normal complex variety M containing $T \cong (\mathbb{C}^*)^n$ as a dense open subset, such that the action of T on itself extends to M.

Example

Fan Σ in $\mathbb{R}^2$



Dual cones in $\mathbb{R}^2$



$$\begin{aligned} M_{\sigma_i} &= \operatorname{Spec} \mathbb{C}[\sigma_i^\vee \cap \mathbb{Z}^2] \\ &\cong \mathbb{C}^2, \\ M_{\rho_i} &= \operatorname{Spec} \mathbb{C}[\rho_i^\vee \cap \mathbb{Z}^2] \\ &\cong \mathbb{C} \times \mathbb{C}^*, \\ M_0 &= \operatorname{Spec} \mathbb{C}[\mathbb{R}^2 \cap \mathbb{Z}^2] \\ &\cong \mathbb{C}^* \times \mathbb{C}^*. \end{aligned}$$

$$M_\Sigma = \cup_{\sigma \in \Sigma} M_\sigma = \mathbb{CP}^2.$$

Theorem (Demazure'70, Kempf–Knudsen–Mumford–Saint-Donat'73, Miyake–Oda'75)

$$\{\textit{fans in } \mathbb{R}^n\} \overset{1-1}{\longleftrightarrow} \{n\text{-dim}_{\mathbb{C}}.\ \textit{toric varieties}\}$$

Cohomological rigidity problem of toric manifolds

A **toric manifold** is a complete smooth toric variety. That is, $|\Sigma| = \mathbb{R}^n$, and each maximal cone has **primitive** ray generators being a basis of $\mathbb{Z}^n$,

$$\det(\rho_{i_1}, \ldots, \rho_{i_n}) = \pm 1 \quad \text{if} \quad \operatorname{cone}(\rho_{i_1}, \ldots, \rho_{i_n}) \in \Sigma_{\max}$$

Theorem (Masuda'08, Higashitani–Kurimoto–Masuda'22)

Let M be a toric manifold.

$$H^*_T(M),\ c_1^T(M) \Longrightarrow \text{ variety isomorphism calss of } M.$$

Problem (Masuda–Suh'08)

Let M and M' be two toric manifolds.

$$H^*(M) \cong H^*(M') \overset{?}{\Longrightarrow} M \cong M' \text{ diffeo (or homeo, hmtp eq).}$$

Quasitoric manifolds

If Σ is the normal fan of a polytope Q, we get **quasitoric manifolds***

$$M_{Q,\rho} := \frac{Q \times (S^1)^n}{\sim_\rho} =_{\text{Set}} \bigcup_{q \in Q} \{q\} \times \frac{(S^1)^n}{\langle \rho_i : \rho_i \text{ “normal” to } q \rangle},$$

by choosing proper ρ_i's satisfying the determinant condition.

Example

$M_{\triangle^2,\rho}$ can be $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$ when $\rho = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$ or $\rho = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

$M_{\triangle^n,\rho} \cong \mathbb{CP}^n$ equivariantly diffeo (Buchstaber–Panov 2000).

Problem (Masuda–Suh'08)

Let M and M' be two quasitoric manifolds.

$$H^*(M) \cong H^*(M') \overset{?}{\Longrightarrow} M \cong M' \text{ diffeo (or homeo, hmtp eq).}$$

$\star$ M. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 62 (1991).

Positive examples of (quasi)toric manifolds

- $b_2 = 2$ (Choi–Park'17).
- **Pogorelov polytopes** (Buchstaber–Erokhovets–Masuda–Panov–Park'17).
- **Fano ones** with $\dim_{\mathbb{C}} \leq 4$ or $b_2 \geq 2\dim_{\mathbb{C}} -2$ (Higashitani–Kurimoto–Masuda'22, Choi'26).
- **Flag fans** under the **separable circuit condition** (Fan–Ma–Wang'24).
- **Bott manifolds** (Choi–Hwang–Jang'25), **blowups** via **one vertex cut** of cubes (Hasui–Kuwata–Masuda–Park'20).
- **Products of simplices** (Fu–So–Song–Theriault'26). Localized hmtp eq.
- ...

Cohomology rings of toric manifolds I

By [Danilov'78, Jurkiewicz'80], $H^*(M_\Sigma)$ is torsion-free.

$$H^*(M_\Sigma) \cong \frac{\mathbb{Z}[D_\rho \mid \text{primitive ray generators } \rho]}{\mathcal{I}_\Sigma + \mathcal{J}_\Sigma}, \qquad \deg D_\rho = 2.$$

- $\mathcal{I}_\Sigma$ is the **Stanley–Reisner ideal**, recording cones not in Σ,
- $\mathcal{J}_\Sigma$ is generated by the linear relations

$$\sum_\rho \langle u, \rho \rangle D_\rho \,, \qquad u \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{Z}).$$

Remark:

- $b_{2k} = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} \#\Sigma(n-i)$. In particular, $\sum_{k=0}^{2n} b_k = \#\Sigma(n)$.
- $\int_{M_\Sigma} D_{i_1} \cdots D_{i_n} = 1$, if $\mathrm{cone}(\rho_{i_1}, \cdots, \rho_{i_n}) \in \Sigma(n)$.

Cohomology rings of toric manifolds II

Suppose that $H^2(M)$ has basis $e_1, \ldots, e_r$. Define the **intersection form**

$$P_M(\mathbf{a}) := \int_M (a_1 e_1 + \cdots + a_r e_r)^n.$$

Let $\mathrm{Sym}(H^2(M))$ act on $\mathbb{Z}[a_1, \ldots, a_r]$ by $e_i \cdot f = \partial f / \partial a_i$.

Theorem (Khovanskii–Pukhlikov'77)

Define $\mathrm{Ann}(P_M) := \{F \in \mathit{Sym}(H^2(M)) : F(\partial) P_M = 0\}$. *Then*

$$H^*(M) \cong \frac{\mathit{Sym}(H^2(M))}{\mathrm{Ann}(P_M)}.$$

Consequently, $U \in \mathrm{GL}(H^2(M))$ extends to $\mathrm{Aut}(H^*(M))$ if and only if

$$P_M(U\mathbf{a}) = \tau P_M(\mathbf{a}) \quad \text{for some } \tau \in \mathbb{Z}^\times.$$

Counterexample: two toric four-folds

$$\Sigma_{(\mathbb{CP}^1)^3} = (\longleftrightarrow)^{\times 3} \Longrightarrow \rho_{(\mathbb{CP}^1)^3} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad \#\Sigma(3) = 8$$

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Star subdivisions $\Big\Vert$ $\rho_7 \hookrightarrow \langle \rho_1, \rho_3, \rho_5 \rangle$, $\rho_8 \hookrightarrow \langle \rho_5, \rho_7 \rangle$, $\rho_9 \hookrightarrow \langle \rho_3, \rho_5 \rangle$

$$\rho_B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 2 & 1 \end{pmatrix} \qquad \#\Sigma_B(3) = 14$$

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$\langle \rho_i, \rho_j, \rho_k \rangle$

$\langle \tilde{\rho}_i, \tilde{\rho}_j, \tilde{\rho}_k, \rho_{10} \rangle$, $\langle \tilde{\rho}_i, \tilde{\rho}_j, \tilde{\rho}_k, \rho_{11} \rangle$

$$\rho_X = \left(\begin{array}{cccccccccc|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 2 & 5 & 3 & 1 & -1 \end{array}\right) \qquad \#\Sigma_X(4) = 28$$

$$\rho_Y = \left(\begin{array}{cccccccccc|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 & -2 & -3 & 1 & 1 & -1 \end{array}\right) \qquad \#\Sigma_Y(4) = 28$$

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$$\rho_B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 2 & 1 \end{pmatrix} \qquad \#\Sigma_B(3) = 14$$

$\langle \rho_i, \rho_j, \rho_k \rangle$

$\langle \tilde{\rho}_i, \tilde{\rho}_j, \tilde{\rho}_k, \rho_{10} \rangle$, $\langle \tilde{\rho}_i, \tilde{\rho}_j, \tilde{\rho}_k, \rho_{11} \rangle$

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Theorem

$H^*(X) \cong H^*(Y)$ *but* $X \not\cong Y$.

Cohomology isomorphism

In $H^2(B) \cong \{\text{algebraic line bundles over } B\}/\sim$, define

$$u := D_2 - D_4 + D_6 - 2D_7 - 4D_8 - D_9,$$
$$v := -D_2 - D_4 + D_6 + D_8 + 2D_9.$$

Then $\quad X = \mathbb{P}(L_u \oplus L_v), \qquad Y = \mathbb{P}(\underline{\mathbb{C}} \oplus L_{u+v}), \qquad uv = 0 \in H^4(B).$

Proposition

Let $z_X := c_1(S_X^*)$, $z_Y := c_1(S_Y^*)$, with S_X the **tautological line bundle** over X, and S_X^* its **dual bundle**.

$$H^*(X) \cong \frac{H^*(B)[z_X]}{\langle z_X^2 + c_1(L_u \oplus L_v) z_X + c_2(L_u \oplus L_v)\rangle} = \frac{H^*(B)[z_X]}{\langle z_X^2 + (u+v) z_X\rangle},$$

$$H^*(Y) \cong \frac{H^*(B)[z_Y]}{\langle z_Y^2 + (u+v) z_Y\rangle}.$$

The isomorphism $z_X \mapsto z_Y$ preserves the total Chern class.

Strategy of proving homotopy non-equivalence

1. **A bundle with a section.** Construct a rank-two vector bundle Θ_X over X admitting a nowhere-zero section.
2. **An obstruction over B.** Show that, for every odd integer n,

$$E_n := L_{nu} \oplus L_{nv}$$

has no nowhere-zero section, using **Rohlin's theorem**.

3. **Control of cohomology ring isomorphisms.**
For every ring isomorphism $\Phi : H^*(X) \to H^*(Y)$, prove that

$$\{\Phi(z_X + u), \Phi(z_X + v)\} = \{\varepsilon(z_Y + u), \varepsilon(z_Y + v)\}$$

for some $\varepsilon \in \{\pm 1\}$.

4. **Derive a contradiction.** If $h\colon\ Y \xrightarrow{\simeq} X$ exists, pulling back Θ_X along

$$B \xrightarrow{j} Y \xrightarrow{h} X$$

produces E_ε with a nowhere-zero section, contradicting Step 2.

Bundles and sections

Define $\Theta_X := \pi_X^*(L_u \oplus L_v) \otimes S_X^* \cong L_{z_X+u} \oplus L_{z_X+v}$.
The inclusion $S_X \hookrightarrow \pi_X^*(L_u \oplus L_v)$ gives a **nowhere-zero section**.

Proposition

Bundle $E_n := L_{nu} \oplus L_{nv}$ has no nowhere-zero section for odd n.

Proof. If (s_1, s_2) were such a section, with $s_1^{-1}(0) \cap s_2^{-1}(0) = \emptyset$.
Then $D := s_1^{-1}(0)$ would be a closed 4-$\dim_{\mathbb{R}}$ manifold, with $L_{nv}|_D$ trivial.

$$TD \oplus L_{nu}|_D = TB|_D, \quad nu \cap [B] = i_*[D].$$

Since $w_1(D) = w_2(D) = 0$, D is spin. Hirzebruch signature theorem gives

$$\sigma(D) = \frac{\int_D p_1(D)}{3} = \frac{\int_B nu\big(p_1(B) - n^2u^2\big)}{3} = \frac{8n(4n^2-1)}{3} \equiv 8 \pmod{16}.$$

Contradiction to **Rohlin's theorem**, which requires $16 \mid \sigma(D)$.

Intersection forms

Use basis $e = (D_2, D_4, D_6, D_7 + D_8, D_8, D_9)$ of $H^2(B)$, let $\mathbf{a} = \sum_{i=1}^{6} a_i e_i$.

$$P_B(\mathbf{a}) = \int_B \mathbf{a}^3 = a_1(6a_2a_3 - 3a_6^2) + a_4^3 + 3a_4a_5^2 - 3a_4a_6^2 - 3a_5a_6^2 + 3a_6^3.$$

Set $W := X$, Y and $\zeta_W := z_W + \frac{u+v}{2}$. Since $H^2(W;\mathbb{Q}) = H^2(B;\mathbb{Q}) \oplus \mathbb{Q}\zeta$,

$$P_W(\mathbf{a}, b) = \int_W (\mathbf{a} + b\zeta_W)^4 = b\big(4P_B(\mathbf{a}) + b^2 \int_B (u+v)^2\mathbf{a}\big),$$

Key properties

1. P_B is irreducible over $\mathbb{Q}$.
2. Suppose $P_B(U\mathbf{a}) = \tau P_B(\mathbf{a})$ for some $U \in \mathrm{GL}_6(\mathbb{Z})$ and $\tau \in \mathbb{Z}^\times$. Then $U(v-u) = \tau(v-u)$.
3. b is the unique rational linear factor of P_W. Hence every graded ring isomorphism $\Phi\colon H^*(X) \cong H^*(Y)$ preserves $H^*(B)$.

Control of cohomology ring isomorphisms

An isomorphism $\Phi : H^*(X;\mathbb{Z}) \xrightarrow{\cong} H^*(Y;\mathbb{Z})$ preserves $H^*(B;\mathbb{Z})$. Then

$$P_B(\Phi\mathbf{a}) = \tau P_B(\mathbf{a}), \quad \Phi(v-u) = \tau(v-u), \quad \tau = \pm 1.$$

The image of $\zeta_X = z_X + \frac{u+v}{2}$ is

$$\Phi(\zeta_X) = \varepsilon\zeta_Y + h, \quad \varepsilon = \pm 1, \quad h \in H^2(B;\mathbb{Q}).$$

As $\Phi(\zeta_X^2) \in H^4(B;\mathbb{Q})$, comparing the coefficient of ζ_Y gives $h = 0$.
Since $z_W + u = \zeta_W - \frac{v-u}{2}$ and $z_W + v = \zeta_W + \frac{v-u}{2}$, we obtain

$$\Phi(z_X + u) = \varepsilon\zeta_Y - \frac{\tau}{2}(v-u),$$
$$\Phi(z_X + v) = \varepsilon\zeta_Y + \frac{\tau}{2}(v-u).$$

Proposition

$$\{\Phi(z_X+u), \Phi(z_X+v)\} = \{\varepsilon(z_Y+u), \varepsilon(z_Y+v)\}.$$

Homotopy non-equivalence

The trivial summand of $Y = \mathbb{P}(\underline{\mathbb{C}} \oplus L_{u+v})$ gives a section

$$j : B \to Y \text{ with } j^*(z_Y) = 0.$$

The pullback of $\Theta_X = L_{z_X+u} \oplus L_{z_X+v}$ along $B \xrightarrow{j} Y \xrightarrow[\simeq]{h} X$ has

$$\{z_X + u, z_X + v\} \xrightarrow{h^*} \{\epsilon(z_Y + u), \epsilon(z_Y + v)\} \xrightarrow{j^*} \{\epsilon u, \epsilon v\}.$$

Hence

$$(hj)^*\Theta_X \cong L_{\varepsilon u} \oplus L_{\varepsilon v} = E_\varepsilon.$$

Contradiction

Bundle Θ_X has a nowhere-zero section, so does $(hj)^*\Theta_X$, but E_ε does not.

Therefore, X and Y are **not homotopy equivalent**.

Generalization

Define $\mathrm{SZ}^{2,2}(-)$ to be the image of

$$[-, K(\mathbb{Z},2) \vee K(\mathbb{Z},2)] \to [-, K(\mathbb{Z},2) \times K(\mathbb{Z},2)].$$

Let B be a closed oriented manifold with $\dim_{\mathbb{R}} = 6$ whose cohomology ring is generated by $H^2(B;\mathbb{Z})$. Let $u, v \in H^2(B)$ with $uv = 0$ satisfy:

1. $(u, v) \notin \mathrm{SZ}^{2,2}(B)$;
2. $(u+v)^2 \neq 0$;
3. there exists $h \in H^2(B)$ such that $(u+v)^2 h = 0$ and $h^3 \neq 0$;
4. every graded ring automorphism of $H^*(B)$ that fixes $(u-v)^2$ sends $u - v$ to $\pm(u-v)$.

Then, for every $N = \mathbb{CP}^{d_1} \times \cdots \times \mathbb{CP}^{d_k}$, the smooth manifolds

$$\mathbb{P}(L_u \oplus L_v) \times N \not\simeq \mathbb{P}(\underline{\mathbb{C}} \oplus L_{u+v}) \times N$$

but they have isomorphic integral cohomology rings.

Problems

Problem (Masuda–Suh'08)

Let M and M' be two toric manifolds with $\dim_{\mathbb{C}} = 3$.

$$H^*(M) \cong H^*(M') \overset{?}{\Longrightarrow} M \cong M' \text{ diffeo (or homeo, hmtp eq).}$$

Homotopical rigidity problem for toric manifolds (Masuda'10)

Let M and M' be two toric manifolds.

$$M \simeq M' \overset{?}{\Longrightarrow} M \cong M' \text{ diffeo (or homeo).}$$

Thank you!!