The Kahler Degrees of Self-Maps of Grassmannians

Haibao Duan

Institute of Mathematics, Chinese Academy of Sciences

Greater Bay Area Topology Conference 2026, November 21-23

Plan of the talk

1. The Problem and its background;
2. Early results (1973-2009);
3. Main results (jointly with Ruizhi Huang and Xuezhi Zhao);
4. Two related questions.

1. The Problem and its background

Let G be a compact connected semi-simple Lie group with Lie algebra $L(G)$, exponential map

$$\exp : L(G) \to G.$$

1. The Problem and its background

Let G be a compact connected semi-simple Lie group with Lie algebra $L(G)$, exponential map

$$\exp : L(G) \to G.$$

Fix a maximal torus $T \subset G$, and let

$$\Omega = \{\omega_1, \cdots, \omega_n\} \subset L(T), \ \ n = \dim T,$$

be a set of fundamental dominant weights.

1. The Problem and its background

Let G be a compact connected semi-simple Lie group with Lie algebra $L(G)$, exponential map

$$\exp : L(G) \to G.$$

Fix a maximal torus $T \subset G$, and let

$$\Omega = \{\omega_1, \cdots, \omega_n\} \subset L(T), \ \ n = \dim T,$$

be a set of fundamental dominant weights. For each weight $\omega_i \in \Omega$ let P_i denote the centralizer of the 1-parameter subgroup

$$\{\exp(t\omega_i) \in G \mid t \in \mathbb{R}\}.$$

1. The Problem and its background

Let G be a compact connected semi-simple Lie group with Lie algebra $L(G)$, exponential map

$$\exp : L(G) \to G.$$

Fix a maximal torus $T \subset G$, and let

$$\Omega = \{\omega_1, \cdots, \omega_n\} \subset L(T), \ \ n = \dim T,$$

be a set of fundamental dominant weights. For each weight $\omega_i \in \Omega$ let P_i denote the centralizer of the 1-parameter subgroup

$$\{\exp(t\omega_i) \in G \mid t \in \mathbb{R}\}.$$

Definition. The homogeneous G/P_i is a complex projective manifold, called the **generalized Grassmannian** of G associated with the weight ω_i.

1. The Problem and its background

Let G be a compact connected semi-simple Lie group with Lie algebra $L(G)$, exponential map

$$\exp : L(G) \to G.$$

Fix a maximal torus $T \subset G$, and let

$$\Omega = \{\omega_1, \cdots, \omega_n\} \subset L(T), \ \ n = \dim T,$$

be a set of fundamental dominant weights. For each weight $\omega_i \in \Omega$ let P_i denote the centralizer of the 1-parameter subgroup

$$\{\exp(t\omega_i) \in G \mid t \in \mathbb{R}\}.$$

Definition. The homogeneous G/P_i is a complex projective manifold, called the **generalized Grassmannian** of G associated with the weight ω_i.

Corollary (Classification). For each simple Lie group G of rank n, there are precisely n generalized Grassmannians.

1. The Problem and its background

Examples. Generalized Grassmannians provide a rich family of fundamental complex projective manifolds of classical interests.

1. The Problem and its background

Examples. Generalized Grassmannians provide a rich family of fundamental complex projective manifolds of classical interests.

- The Grassmannian $G_{n,k} = \frac{U(n)}{U(k)\times U(n-k)}$ of k-linear subspaces L of $\mathbb{C}^n$;

1. The Problem and its background

Examples. Generalized Grassmannians provide a rich family of fundamental complex projective manifolds of classical interests.

- The Grassmannian $G_{n,k} = \frac{U(n)}{U(k)\times U(n-k)}$ of k-linear subspaces L of $\mathbb{C}^n$;
- The Grassmannian $\mathbb{C}S_n = \frac{SO(2n)}{U(n)}$ of complex structures on $\mathbb{R}^{2n}$:

$$\mathbb{C}S_n = \{J \in SO(2n) \mid J^2 = -I_{2n}.\}$$

1. The Problem and its background

Examples. Generalized Grassmannians provide a rich family of fundamental complex projective manifolds of classical interests.

- The Grassmannian $G_{n,k} = \frac{U(n)}{U(k)\times U(n-k)}$ of k-linear subspaces L of $\mathbb{C}^n$;
- The Grassmannian $\mathbb{C}S_n = \frac{SO(2n)}{U(n)}$ of complex structures on $\mathbb{R}^{2n}$:

$$\mathbb{C}S_n = \{J \in SO(2n) \mid J^2 = -I_{2n}.\}$$

- Eight generalized Grassmannians G/P associated with the exceptional Lie groups:

G	F_4	F_4	E_6	E_6	E_7	E_7	E_7	E_8
ω	ω_1	ω_4	ω_2	ω_6	ω_1	ω_7	ω_2	ω_8
P	$C_3 \cdot S^1$	$B_3 \cdot S^1$	$A_6 \cdot S^1$	$D_5 \cdot S^1$	$D_6 \cdot S^1$	$E_6 \cdot S^1$	$A_7 \cdot S^1$	$E_7 \cdot S^1$

1. The Problem and its background

Theorem. The integral cohomology of a generalized Grassmannian $X = G/P_i$ has the following properties.

[1] R. Bott and H. Samelson, Application of the theory of Morse to symmetric spaces, Amer. J. Math.(1958).

[2] H. Duan, X. Zhao, The Chow rings of generalized Grassmannians, Found. Math. Comput, 2010.

1. The Problem and its background

Theorem. The integral cohomology of a generalized Grassmannian $X = G/P_i$ has the following properties.

1. The cohomology $H^r(X)$ is torsion free and vanishes for odd r [1];

[1] R. Bott and H. Samelson, Application of the theory of Morse to symmetric spaces, Amer. J. Math.(1958).

[2] H. Duan, X. Zhao, The Chow rings of generalized Grassmannians, Found. Math. Comput, 2010.

1. The Problem and its background

Theorem. The integral cohomology of a generalized Grassmannian $X = G/P_i$ has the following properties.

1. The cohomology $H^r(X)$ is torsion free and vanishes for odd r [1];
2. $H^2(X) = \mathbb{Z}$ is generated by an integral Kahler class κ_X of X;

[1] R. Bott and H. Samelson, Application of the theory of Morse to symmetric spaces, Amer. J. Math.(1958).

[2] H. Duan, X. Zhao, The Chow rings of generalized Grassmannians, Found. Math. Comput, 2010.

1. The Problem and its background

Theorem. The integral cohomology of a generalized Grassmannian $X = G/P_i$ has the following properties.

1. The cohomology $H^r(X)$ is torsion free and vanishes for odd r [1];
2. $H^2(X) = \mathbb{Z}$ is generated by an integral Kahler class κ_X of X;
3. The ring $H^*(X)$ can be constructed from the Cartan matrix of G [2]

[1] R. Bott and H. Samelson, Application of the theory of Morse to symmetric spaces, Amer. J. Math.(1958).

[2] H. Duan, X. Zhao, The Chow rings of generalized Grassmannians, Found. Math. Comput, 2010.

1. The Problem and its background

Theorem. The integral cohomology of a generalized Grassmannian $X = G/P_i$ has the following properties.

1. The cohomology $H^r(X)$ is torsion free and vanishes for odd r [1];
2. $H^2(X) = \mathbb{Z}$ is generated by an integral Kahler class κ_X of X;
3. The ring $H^*(X)$ can be constructed from the Cartan matrix of G [2]

Definition. For each self-map $f : X \to X$, there exists an integer $deg(f) \in \mathbb{Z}$, called the **Kahler degree** of f, characterized uniquely by

$$f^*(\kappa_X) = deg(f) \cdot \kappa_X.$$

[1] R. Bott and H. Samelson, Application of the theory of Morse to symmetric spaces, Amer. J. Math.(1958).

[2] H. Duan, X. Zhao, The Chow rings of generalized Grassmannians, Found. Math. Comput, 2010.

1. The Problem and its background

Theorem. The integral cohomology of a generalized Grassmannian $X = G/P_i$ has the following properties.

1. The cohomology $H^r(X)$ is torsion free and vanishes for odd r [1];
2. $H^2(X) = \mathbb{Z}$ is generated by an integral Kahler class κ_X of X;
3. The ring $H^*(X)$ can be constructed from the Cartan matrix of G [2]

Definition. For each self-map $f : X \to X$, there exists an integer $deg(f) \in \mathbb{Z}$, called the **Kahler degree** of f, characterized uniquely by

$$f^*(\kappa_X) = deg(f) \cdot \kappa_X.$$

For an integer $k \in \mathbb{Z}$, the **Adams operator** $\psi^k \in \mathrm{End}(H^*(X))$ is well defined by

$$\psi^k(x) = k^r x, \ x \in H^{2r}(X).$$

[1] R. Bott and H. Samelson, Application of the theory of Morse to symmetric spaces, Amer. J. Math.(1958).

[2] H. Duan, X. Zhao, The Chow rings of generalized Grassmannians, Found. Math. Comput, 2010.

1. The Problem and its background

For a topological space X let $[X, X]$ denote the monoid of homotopy classes of self-maps of X, and let $\mathrm{End}(H^*(X))$ denote the monoid of endomorphisms of the integral cohomology ring $H^*(X)$.

[3] D. Sullivan, Infinitesimal Computations in Topology, Publications Mathmatiques de l'IHS, 1977.

1. The Problem and its background

For a topological space X let $[X, X]$ denote the monoid of homotopy classes of self-maps of X, and let $\mathrm{End}(H^*(X))$ denote the monoid of endomorphisms of the integral cohomology ring $H^*(X)$. Consider the representation

$$R : [X, X] \to \mathrm{End}(H^*(X)), \quad [f] \to f^*.$$

[3] D. Sullivan, Infinitesimal Computations in Topology, Publications Mathmatiques de l'IHS, 1977.

1. The Problem and its background

For a topological space X let $[X, X]$ denote the monoid of homotopy classes of self-maps of X, and let $\mathrm{End}(H^*(X))$ denote the monoid of endomorphisms of the integral cohomology ring $H^*(X)$. Consider the representation

$$R : [X, X] \to \mathrm{End}(H^*(X)), \quad [f] \to f^*.$$

It happens that, if $X = G/P_i$ is a generalized Grassmannian, then the representation R is faithful up to finite ambiguity. Precisely, we have

[3] D. Sullivan, Infinitesimal Computations in Topology, Publications Mathmatiques de l'IHS, 1977.

1. The Problem and its background

For a topological space X let $[X, X]$ denote the monoid of homotopy classes of self-maps of X, and let $\mathrm{End}(H^*(X))$ denote the monoid of endomorphisms of the integral cohomology ring $H^*(X)$. Consider the representation

$$R : [X, X] \to \mathrm{End}(H^*(X)), \quad [f] \to f^*.$$

It happens that, if $X = G/P_i$ is a generalized Grassmannian, then the representation R is faithful up to finite ambiguity. Precisely, we have

Theorem (Sullivan, 1977)[3] Let X_0 denote the rationalization of a generalized Grassmannian $X = G/P_i$. Then R induces an isomorphism of monoids:

$$[X_0, X_0] \stackrel{\cong}{\to} \mathrm{End}(H^*(X; \mathbb{Q})), \ [f] \to f^*.$$

[3] D. Sullivan, Infinitesimal Computations in Topology, Publications Mathmatiques de l'IHS, 1977.

1. The Problem and its background

This motivates the main problem of this talk.

Problem 1. For each generalized Grassmannian $X = G/P_i$, determine the representation

$$R : [X, X] \to \mathrm{End}(H^*(X)), \quad [f] \to f^*$$

explicitly.

2. Early results

Let γ be the canonical k dimensional complex vector bundle over $G_{n,k}$, and let

$$C(\gamma) = 1 + c_1 + \cdots + c_k, \ c_r \in H^{2r}(G_{n,k})$$

denote the total Chern class of γ.

[4] Borel A., Sur la cohomologie des espaces fibrs principaux et des espaces homogenes de groupes de Lie compacts, Ann. Math. 1953.

[5] M. Hoffman, Endomorphisms of the cohomology of complex Grassmannians, Trans. AMS, 1984.

2. Early results

Let γ be the canonical k dimensional complex vector bundle over $G_{n,k}$, and let

$$C(\gamma) = 1 + c_1 + \cdots + c_k, \ c_r \in H^{2r}(G_{n,k})$$

denote the total Chern class of γ. Borel [4] has shown that

$$H^*(G_{n,k}) = \mathbb{Z}[c_1, \cdots, c_k]/I_{n,k}, \ I_{n,k} = \langle \bar{c}_{n-k+1}, \cdots, \bar{c}_n \rangle,$$

where $\bar{c}_i$ is the degree i component of the formal inverse

$$C(\gamma)^{-1} = 1 + \bar{c}_1 + \cdots + \bar{c}_m + \cdots .$$

[4] Borel A., Sur la cohomologie des espaces fibrs principaux et des espaces homogenes de groupes de Lie compacts, Ann. Math. 1953.

[5] M. Hoffman, Endomorphisms of the cohomology of complex Grassmannians, Trans. AMS, 1984.

2. Early results

Let γ be the canonical k dimensional complex vector bundle over $G_{n,k}$, and let

$$C(\gamma) = 1 + c_1 + \cdots + c_k, \ c_r \in H^{2r}(G_{n,k})$$

denote the total Chern class of γ. Borel [4] has shown that

$$H^*(G_{n,k}) = \mathbb{Z}[c_1, \cdots, c_k]/I_{n,k}, \ I_{n,k} = \langle \bar{c}_{n-k+1}, \cdots, \bar{c}_n \rangle,$$

where $\bar{c}_i$ is the degree i component of the formal inverse

$$C(\gamma)^{-1} = 1 + \bar{c}_1 + \cdots + \bar{c}_m + \cdots .$$

Theorem 1 (Brewste(1978)-Hoffman(1984)[5] Let $f : G_{n,k} \to G_{n,k}$ be a self-map with nonzero Kahler degree $deg(f) = m$.

[4] Borel A., Sur la cohomologie des espaces fibrs principaux et des espaces homogenes de groupes de Lie compacts, Ann. Math. 1953.

[5] M. Hoffman, Endomorphisms of the cohomology of complex Grassmannians, Trans. AMS, 1984.

2. Early results

Let γ be the canonical k dimensional complex vector bundle over $G_{n,k}$, and let

$$C(\gamma) = 1 + c_1 + \cdots + c_k, \ c_r \in H^{2r}(G_{n,k})$$

denote the total Chern class of γ. Borel [4] has shown that

$$H^*(G_{n,k}) = \mathbb{Z}[c_1, \cdots, c_k]/I_{n,k}, \ I_{n,k} = \langle \bar{c}_{n-k+1}, \cdots, \bar{c}_n \rangle,$$

where $\bar{c}_i$ is the degree i component of the formal inverse

$$C(\gamma)^{-1} = 1 + \bar{c}_1 + \cdots + \bar{c}_m + \cdots .$$

Theorem 1 (Brewste(1978)-Hoffman(1984)[5] Let $f : G_{n,k} \to G_{n,k}$ be a self-map with nonzero Kahler degree $deg(f) = m$.

- If $n \neq 2k$, then $f^* = \psi^m$.

[4] Borel A., Sur la cohomologie des espaces fibrs principaux et des espaces homogenes de groupes de Lie compacts, Ann. Math. 1953.

[5] M. Hoffman, Endomorphisms of the cohomology of complex Grassmannians, Trans. AMS, 1984.

2. Early results

Let γ be the canonical k dimensional complex vector bundle over $G_{n,k}$, and let

$$C(\gamma) = 1 + c_1 + \cdots + c_k, \ c_r \in H^{2r}(G_{n,k})$$

denote the total Chern class of γ. Borel [4] has shown that

$$H^*(G_{n,k}) = \mathbb{Z}[c_1, \cdots, c_k]/I_{n,k}, \ I_{n,k} = \langle \bar{c}_{n-k+1}, \cdots, \bar{c}_n \rangle,$$

where $\bar{c}_i$ is the degree i component of the formal inverse

$$C(\gamma)^{-1} = 1 + \bar{c}_1 + \cdots + \bar{c}_m + \cdots .$$

Theorem 1 (Brewste(1978)-Hoffman(1984)[5] Let $f : G_{n,k} \to G_{n,k}$ be a self-map with nonzero Kahler degree $deg(f) = m$.

- If $n \neq 2k$, then $f^* = \psi^m$.
- If $n = 2k$, there is the additional possibility $f^* = \psi^m \circ \tau$, where

$$\tau(c_i) = \bar{c}_i, \ 1 \leq i \leq k.$$

[4] Borel A., Sur la cohomologie des espaces fibrs principaux et des espaces homogenes de groupes de Lie compacts, Ann. Math. 1953.

[5] M. Hoffman, Endomorphisms of the cohomology of complex Grassmannians, Trans. AMS, 1984.

2. Early results

Let γ be the canonical n dimensional complex vector bundle over $\mathbb{C}S_n$, and let

$$C(\gamma) = 1 + d_1 + \cdots + d_{n-1}, \quad d_r \in H^{2r}(\mathbb{C}S_n),$$

be the **refined** total Chern class of γ.

[6] H. Duan, Self-maps of the Grassmannian of complex structures, Compositio Math, 2002.

2. Early results

Let γ be the canonical n dimensional complex vector bundle over $\mathbb{C}S_n$, and let

$$C(\gamma) = 1 + d_1 + \cdots + d_{n-1}, \quad d_r \in H^{2r}(\mathbb{C}S_n),$$

be the **refined** total Chern class of γ. Then the integral cohomology $H^*(\mathbb{C}S_n)$ is

$$H^*(\mathbb{C}S_n) = \mathbb{Z}[d_1, \cdots, d_{n-1}]/I_n, \ I_n = \langle h_1, \cdots, h_{n-1} \rangle,$$

where

$$h_i = d_i^2 - 2d_{i-1}d_{i+1} + \cdots + (-1)^{i-1}2d_1d_{2i-1} + (-1)^i d_{2i}, \ 1 \leq i \leq n-1.$$

[6] H. Duan, Self-maps of the Grassmannian of complex structures, Compositio Math, 2002.

2. Early results

Let γ be the canonical n dimensional complex vector bundle over $\mathbb{C}S_n$, and let

$$C(\gamma) = 1 + d_1 + \cdots + d_{n-1}, \quad d_r \in H^{2r}(\mathbb{C}S_n),$$

be the **refined** total Chern class of γ. Then the integral cohomology $H^*(\mathbb{C}S_n)$ is

$$H^*(\mathbb{C}S_n) = \mathbb{Z}[d_1, \cdots, d_{n-1}]/I_n, \ I_n = \langle h_1, \cdots, h_{n-1} \rangle,$$

where

$$h_i = d_i^2 - 2d_{i-1}d_{i+1} + \cdots + (-1)^{i-1}2d_1d_{2i-1} + (-1)^i d_{2i}, \ 1 \leq i \leq n-1.$$

Theorem 2 (Duan, 2002)[6] If $f : \mathbb{C}S_n \to \mathbb{C}S_n$ is a self-map with nonzero Kahler degree $deg(f) = m$, then

$$f^* = \psi^m.$$

[6] H. Duan, Self-maps of the Grassmannian of complex structures, Compositio Math, 2002.

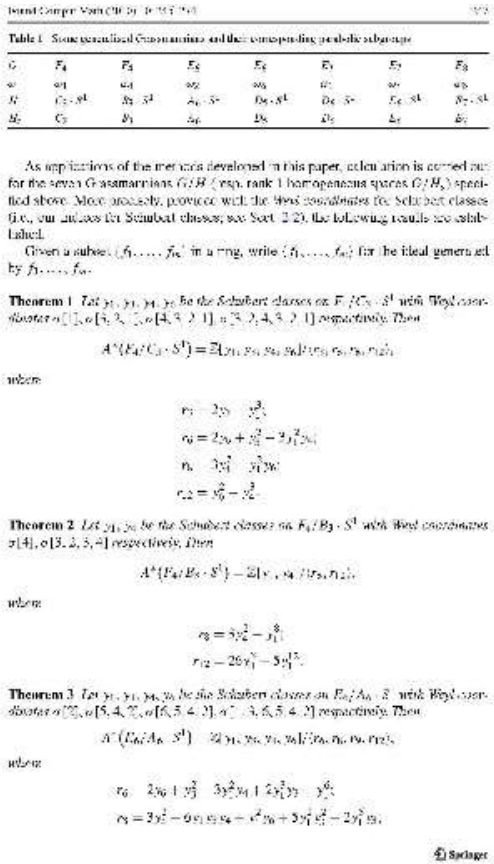
2. Early results

Based on Schubert calculus, Duan-Zhao [7] composed a program to compute the cohomology rings of flag manifolds.

[7] H. Duan, X. Zhao, The Chow rings of generalized Grassmannians, Found. Math. Comput, 2010.

2. Early results

Based on Schubert calculus, Duan-Zhao [7] composed a program to compute the cohomology rings of flag manifolds. As applications, the cohomologies of the eight generalized Grassmannians were computed.



248 Found Comput Math (2010) 10: 245–274

$$r_9 = 2y_3y_6 - y_1^3y_6;$$
$$r_{12} = y_3^3 - y_6^2.$$

Theorem 4 *Let y_1, y_4 be the Schubert classes on $E_6/D_5 \cdot S^1$ with Weyl coordinates $\sigma[6], \sigma[2,4,5,6]$ respectively. Then*

$$A^*(E_6/D_5 \cdot S^1) = \mathbb{Z}[y_1, y_4]/\langle r_9, r_{12}\rangle,$$

where

$$r_9 = 2y_1^9 + 3y_1y_4^2 - 6y_1^5y_4;$$
$$r_{12} = y_4^3 - 6y_1^4y_4^2 + y_1^{12}.$$

Theorem 5 *Let y_1, y_5, y_9 be the Schubert classes on $E_7/E_6 \cdot S^1$ with Weyl coordinates $\sigma[7], \sigma[2,4,5,6,7], \sigma[1,5,4,2,3,4,5,6,7]$ respectively. Then*

$$A^*(E_7/E_6 \cdot S^1) = \mathbb{Z}[y_1, y_5, y_9]/\langle r_{10}, r_{14}, r_{18}\rangle,$$

where

$$r_{10} = y_5^2 - 2y_1y_9;$$
$$r_{14} = 2y_5y_9 - 9y_1^4y_5^2 + 6y_1^9y_5 - y_1^{14};$$
$$r_{18} = y_9^2 + 10y_1^5y_5^3 - 9y_1^8y_5^2 + 2y_1^{13}y_5.$$

Theorem 6 *Let y_1, y_4, y_6, y_9 be the Schubert classes on $E_7/D_6 \cdot S^1$ with Weyl coordinates $\sigma[1], \sigma[2,4,3,1], \sigma[2,6,5,4,3,1], \sigma[3,4,2,7,6,5,4,3,1]$ respectively. Then*

$$A^*(E_7/D_6 \cdot S^1) = \mathbb{Z}[y_1, y_4, y_6, y_9]/\langle r_9, r_{12}, r_{14}, r_{18}\rangle,$$

where

$$r_9 = 2y_9 + 3y_1y_4^2 + 4y_1^3y_6 + 2y_1^5y_4 - 2y_1^9;$$
$$r_{12} = 3y_6^2 - y_4^3 - 3y_1^4y_4^2 - 2y_1^6y_6 + 2y_1^8y_4;$$
$$r_{14} = 3y_4^2y_6 + 3y_1^2y_6^2 + 6y_1^2y_4^3 + 6y_1^4y_4y_6 + 2y_1^5y_9 - y_1^{14};$$
$$r_{18} = 5y_9^2 + 29y_6^3 - 24y_1^6y_6^2 + 45y_1^2y_4y_6^2 + 2y_1^9y_9.$$

Theorem 7 *Let y_1, y_6, y_{10}, y_{15} be the Schubert classes on $E_8/E_7 \cdot S^1$ with Weyl coordinates $\sigma[8], \sigma[3,4,5,6,7,8], \sigma[1,5,4,2,3,4,5,6,7,8], \sigma[5,4,3,1,7,6,5,4,2,3,4,5,6,7,8]$ respectively. Then*

$$A^*(E_8/E_7 \cdot S^1) = \mathbb{Z}[y_1, y_6, y_{10}, y_{15}]/\langle r_{15}, r_{20}, r_{24}, r_{30}\rangle,$$

where

$$r_{15} = 2y_{15} - 16y_1^5y_{10} - 10y_1^3y_6^2 + 10y_1^9y_6 - y_1^{15};$$
$$r_{20} = 3y_{10}^2 + 10y_1^2y_6^3 + 18y_1^4y_6y_{10} - 2y_1^5y_{15} - 8y_1^8y_6^2 + 4y_1^{10}y_{10} - y_1^{14}y_6;$$
$$r_{24} = 5y_6^4 + 30y_1^2y_6^2y_{10} + 15y_1^4y_{10}^2 - 2y_1^9y_{15} - 5y_1^{12}y_6^2 + y_1^{14}y_{10};$$
$$r_{30} = y_{15}^2 - 8y_{10}^3 + y_6^5 - 2y_1^3y_6^2y_{15} + 3y_1^4y_6y_{10}^2 - 8y_1^5y_{10}y_{15} + 6y_1^9y_6y_{15} - 9y_1^{10}y_{10}^2 - y_1^{12}y_6^3 - 2y_1^{14}y_6y_{10} - 3y_1^{15}y_{15} + 8y_1^{20}y_{10} + y_1^{24}y_6 - y_1^{30}.$$

Traditionally, Schubert calculus deals with intersection theory on flag varieties. The algorithms in Sect. 4.3 and the proofs of Theorems 8–14 in Sect. 5 demonstrate how this calculation is extended to homogeneous spaces of other types.

This paper is arranged as follows. Section 2 contains a brief introduction to what we need from Schubert calculus; Section 3 develops some algebraic results concerning computation in the quotient of a polynomial ring. Resorting to the Gysin sequence of circle bundles, the relationship between cohomologies of a Grassmannian G/H and its allied space G/H_δ is formulated in Sect 4. With these preliminaries, Theorems 1–7 (resp. Theorems 8–14) are established in a unified pattern in Sect. 6 (resp. Sect. 5).

Historically, the problem of computing the Chow ring of a flag variety (resp. the integral cohomology of a homogeneous space) has been studied by many authors. Comparisons between our method and the classical means are made in Sect. 7, where mistakes occurring in the earlier computations are corrected in Sect. 7.5.

Certain theoretical notions and results of this paper are also algorithmic in nature. Their effective computability is emphasized by referring to appropriate sections of [13], where intermediate data facilitating our calculations are given in detail. To make the present work self-contained, the most relevant data from [13] are summarized and tabulated in the proofs of Theorems 8–14 in Sect. 5.

2 Elements of Schubert Calculus

Assume throughout that the Lie group G under consideration is compact and 1-connected. Fix a maximal torus T in G and equip the Lie algebra $L(G)$ with an inner product $(\,,\,)$, so that the adjoint representation acts as isometries of $L(G)$. Let $\Phi = \{\beta_1, \ldots, \beta_n\} \subset L(T)$ be a *set of simple roots* of G [20, p. 47]. The *Cartan matrix* of G is $C = (c_{ij})_{n\times n}$, where

$$c_{ij} := 2(\beta_i, \beta_j)/(\beta_j, \beta_j), \quad 1 \le i, j \le n \text{ [20, p. 55]}.$$

We recall two algorithms "*Decomposition*" and "*L–R coefficients*" developed in [11]. The first presents the Weyl group of G by the minimized decompositions of its elements, in terms of which the Schubert varieties on G/H can be constructed. The second expands a polynomial in the Schubert classes as the linear combination of the Schubert basis.

[7] H. Duan, X. Zhao, The Chow rings of generalized Grassmannians, Found. Math. Comput., 2010.

2. Early results

Subsequently, Zhao-Li (fang) composed a program to compute the monoid

$$\mathrm{End}(H^*(X)).$$

[8] Li, Duan, Homology rigidity of Grassmannians, Acta Math. Sci.,2009.

2. Early results

Subsequently, Zhao-Li (fang) composed a program to compute the monoid

$$\mathrm{End}(H^*(X)).$$

As applications, the following result is obtained.

Theorem 3[8] Let X be one of the eight Grassmannians specified in the previous page, and let f be a self map of X with nonzero Kahler degree $deg(f) = m$.

[8] Li, Duan, Homology rigidity of Grassmannians, Acta Math. Sci.,2009.

2. Early results

Subsequently, Zhao-Li (fang) composed a program to compute the monoid

$$\mathrm{End}(H^*(X)).$$

As applications, the following result is obtained.

Theorem 3[8] Let X be one of the eight Grassmannians specified in the previous page, and let f be a self map of X with nonzero Kahler degree $deg(f) = m$.

- If $X \neq E_6/A_6 \cdot S^1$, then $f^* = \psi^m$;

[8] Li, Duan, Homology rigidity of Grassmannians, Acta Math. Sci.,2009.

2. Early results

Subsequently, Zhao-Li (fang) composed a program to compute the monoid

$$\mathrm{End}(H^*(X)).$$

As applications, the following result is obtained.

Theorem 3[8] Let X be one of the eight Grassmannians specified in the previous page, and let f be a self map of X with nonzero Kahler degree $deg(f) = m$.

- If $X \neq E_6/A_6 \cdot S^1$, then $f^* = \psi^m$;
- If $X = E_6/A_6 \cdot S^1$, there is the additional possibility

$$f^* = \psi^m \circ \tau,$$

where

$$\tau(y_1) = y_1, \quad \tau(y_3) = y_1^3 - y_3, \quad \tau(y_4) = y_4 - 2y_1y_3 + y_1^4.$$

[8] Li, Duan, Homology rigidity of Grassmannians, Acta Math. Sci.,2009.

2. Early results

Theorems 1-3 were obtained by a case by case calculation in the integral cohomology rings $H^*(X)$. Therefore, it is impossible to solve Problem 1 by the same method.

2. Early results

Theorems 1-3 were obtained by a case by case calculation in the integral cohomology rings $H^*(X)$. Therefore, it is impossible to solve Problem 1 by the same method.

These bring us to following refinements of Problem 1.

Problem 2. Can we find a unified approach to determine the representation

$$R : [X, X] \to \mathrm{End}(H^*(X)),\ X = G/P_i,$$

uniformly for all G and $P_i \subset G$.

2. Early results

Theorems 1-3 were obtained by a case by case calculation in the integral cohomology rings $H^*(X)$. Therefore, it is impossible to solve Problem 1 by the same method.

These bring us to following refinements of Problem 1.

Problem 2. Can we find a unified approach to determine the representation

$$R : [X, X] \to \mathrm{End}(H^*(X)),\ X = G/P_i,$$

uniformly for all G and $P_i \subset G$.
In particular, apart from the canonical Adams operators

$$\psi^m \in \mathrm{End}(H^*(X)),\ m \in \mathbb{Z},$$

are there other nontrivial endomorphisms $\tau \in \mathrm{End}(H^*(X))$?

3. Main results

The simply connected simple Lie groups G can be classified by their Dynkin diagrams Γ_G,

3. Main results

The simply connected simple Lie groups G can be classified by their Dynkin diagrams Γ_G, which consists of the four infinite classical families

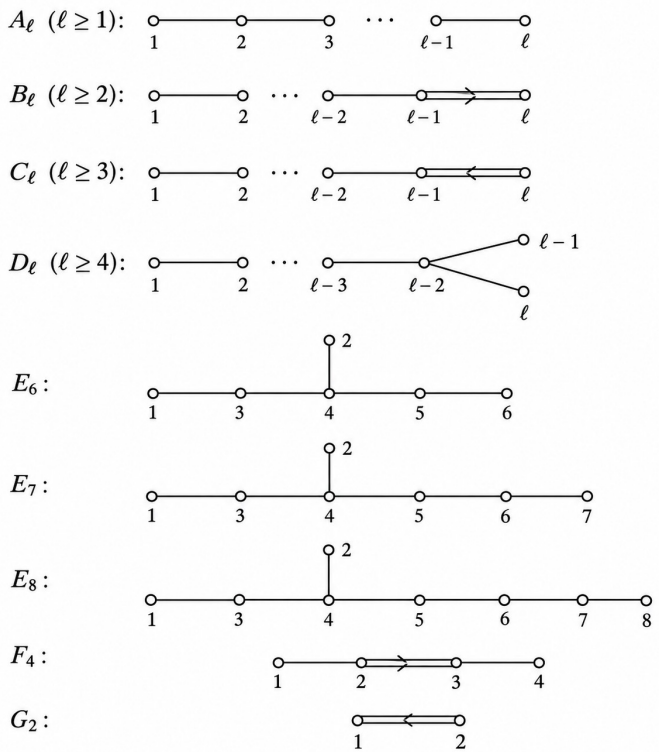
$$\{A_l,\ l \geq 1\}, \{B_l,\ l \geq 2\}, \{C_l,\ l \geq 3\}, \{D_l,\ l \geq 4\},$$

3. Main results

The simply connected simple Lie groups G can be classified by their Dynkin diagrams Γ_G, which consists of the four infinite classical families

$$\{A_l,\ l \geq 1\}, \{B_l,\ l \geq 2\}, \{C_l,\ l \geq 3\}, \{D_l,\ l \geq 4\},$$

together with the five exceptional types G_2, F_4, E_6, E_7, E_8:



3. Main results

Let $\mathrm{Aut}(\Gamma_G)$ denote the automorphism group of the Dynkin diagram Γ_G.

3. Main results

Let $\mathrm{Aut}(\Gamma_G)$ denote the automorphism group of the Dynkin diagram Γ_G. For each vertex $\omega_i \in \Gamma_G$ define its stabilizer by

$$\mathrm{Aut}(\Gamma_G, \omega_i) := \{\sigma \in \mathrm{Aut}(\Gamma_G) \mid \sigma(\omega_i) = \omega_i\}.$$

3. Main results

Let $\mathrm{Aut}(\Gamma_G)$ denote the automorphism group of the Dynkin diagram Γ_G. For each vertex $\omega_i \in \Gamma_G$ define its stabilizer by

$$\mathrm{Aut}(\Gamma_G, \omega_i) := \{\sigma \in \mathrm{Aut}(\Gamma_G) \mid \sigma(\omega_i) = \omega_i\}.$$

A direct **inspection** of the Dynkin diagrams gives the following classification.

Lemma. The stabilizer $\mathrm{Aut}(\Gamma_G, \omega_i)$ is nontrivial if and only if the pair (G, i) appears in the following table.

G	A_{2k+1}	$D_l,\ l > 4$	D_4	E_6
i	$k+1$	$1, \cdots, n-2$	$2, 1, 3, 4$	$2, 4$
$\mathrm{Aut}(\Gamma_G, \omega_i)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	S_3, S_2, S_2, S_2	$\mathbb{Z}_2$

.

Here S_m denotes the symmetric group on m letters.

3. Main results

Every Dynkin diagram automorphism $\sigma \in \mathrm{Aut}(\Gamma_G)$ extends uniquely to a Lie algebra automorphism

$$\theta_\sigma : L(G) \to L(G).$$

3. Main results

Every Dynkin diagram automorphism $\sigma \in \mathrm{Aut}(\Gamma_G)$ extends uniquely to a Lie algebra automorphism

$$\theta_\sigma : L(G) \to L(G).$$

Since G is simply connected, θ_σ integrates uniquely to a Lie group automorphism

$$\Theta_\sigma : G \to G, \ T_e\Theta_\sigma = \theta_\sigma.$$

3. Main results

Every Dynkin diagram automorphism $\sigma \in \mathrm{Aut}(\Gamma_G)$ extends uniquely to a Lie algebra automorphism

$$\theta_\sigma : L(G) \to L(G).$$

Since G is simply connected, θ_σ integrates uniquely to a Lie group automorphism

$$\Theta_\sigma : G \to G, \ \ T_e\Theta_\sigma = \theta_\sigma.$$

Theorem A. For every $\sigma \in \mathrm{Aut}(\Gamma_G, \omega_i)$, the Lie group automorphism Θ_σ descends to a diffeomorphism of the quotient manifold

$$h_\sigma : G/P_i \to G/P_i$$

making the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{\Theta_\sigma} & G \\ \pi \downarrow & & \downarrow \pi \\ G/P_i & \xrightarrow{h_\sigma} & G/P_i \end{array} .$$

3. Main results

Every Dynkin diagram automorphism $\sigma \in \mathrm{Aut}(\Gamma_G)$ extends uniquely to a Lie algebra automorphism

$$\theta_\sigma : L(G) \to L(G).$$

Since G is simply connected, θ_σ integrates uniquely to a Lie group automorphism

$$\Theta_\sigma : G \to G, \ \ T_e\Theta_\sigma = \theta_\sigma.$$

Theorem A. For every $\sigma \in \mathrm{Aut}(\Gamma_G, \omega_i)$, the Lie group automorphism Θ_σ descends to a diffeomorphism of the quotient manifold

$$h_\sigma : G/P_i \to G/P_i$$

making the following diagram commute:

$$\begin{array}{ccc} G & \stackrel{\Theta_\sigma}{\to} & G \\ \pi \downarrow & & \downarrow \pi \\ G/P_i & \stackrel{h_\sigma}{\to} & G/P_i \end{array} .$$

Moreover,

$$\sigma \neq \mathrm{id} \Rightarrow h_\sigma^* \neq \mathrm{Id} \in \mathrm{End}(H^*(X)).$$

3. Main results

Let $X = G/P_{\{i\}}$ be a generalized Grassmannian, and let $\sigma \in \mathrm{Aut}(\Gamma_G, \omega_i)$ be nontrivial.

3. Main results

Let $X = G/P_{\{i\}}$ be a generalized Grassmannian, and let $\sigma \in \mathrm{Aut}(\Gamma_G, \omega_i)$ be nontrivial. The automorphism

$$h_\sigma^* \in \mathrm{Aut}(H^*(X))$$

arising in Theorem A is called a **Dynkin symmetry** of the ring $H^*(X)$.

3. Main results

Let $X = G/P_{\{i\}}$ be a generalized Grassmannian, and let $\sigma \in \mathrm{Aut}(\Gamma_G, \omega_i)$ be nontrivial. The automorphism

$$h_\sigma^* \in \mathrm{Aut}(H^*(X))$$

arising in Theorem A is called a **Dynkin symmetry** of the ring $H^*(X)$.
Our solution to Problem 2 is:

Theorem B. Let f be a self-map of a generalized Grassmannian X with nonzero Kahler degree $deg(f) = m \neq 0$.

3. Main results

Let $X = G/P_{\{i\}}$ be a generalized Grassmannian, and let $\sigma \in \mathrm{Aut}(\Gamma_G, \omega_i)$ be nontrivial. The automorphism

$$h_\sigma^* \in \mathrm{Aut}(H^*(X))$$

arising in Theorem A is called a **Dynkin symmetry** of the ring $H^*(X)$.
Our solution to Problem 2 is:

Theorem B. Let f be a self-map of a generalized Grassmannian X with nonzero Kahler degree $deg(f) = m \neq 0$. Then

$$f^* = \psi^m \text{ or } f^* = \psi^m \circ \tau,$$

3. Main results

Let $X = G/P_{\{i\}}$ be a generalized Grassmannian, and let $\sigma \in \mathrm{Aut}(\Gamma_G, \omega_i)$ be nontrivial. The automorphism

$$h_\sigma^* \in \mathrm{Aut}(H^*(X))$$

arising in Theorem A is called a **Dynkin symmetry** of the ring $H^*(X)$.
Our solution to Problem 2 is:

Theorem B. Let f be a self-map of a generalized Grassmannian X with nonzero Kahler degree $deg(f) = m \neq 0$. Then

$$f^* = \psi^m \text{ or } f^* = \psi^m \circ \tau,$$

where the second possibility can occur only if $\mathrm{Aut}(\Gamma_G, \omega_i) \neq \{\mathrm{id}\}$, and τ is a Dynkin symmetry of the ring $H^*(X)$ classified in the previous lemma.

3. Main results

Theorems A and B has two direct consequence.

3. Main results

Theorems A and B has two direct consequence.

Corollary 1. The homotopy mapping class group of a generalized Grassmannian X is

$$\mathrm{Aut}(X) = \{f \in [X, X] \mid deg(f) = \pm 1\}.$$

3. Main results

Theorems A and B has two direct consequence.

Corollary 1. The homotopy mapping class group of a generalized Grassmannian X is

$$\mathrm{Aut}(X) = \{f \in [X, X] \mid deg(f) = \pm 1\}.$$

Corollary 2. Let $X = G/P_{\{i\}}$ be a generalized Grassmannian with G of exceptional Lie type, and suppose that

$$(G, i) \neq (E_6, 2), \quad (E_6, 4).$$

3. Main results

Theorems A and B has two direct consequence.

Corollary 1. The homotopy mapping class group of a generalized Grassmannian X is

$$\mathrm{Aut}(X) = \{f \in [X, X] \mid deg(f) = \pm 1\}.$$

Corollary 2. Let $X = G/P_{\{i\}}$ be a generalized Grassmannian with G of exceptional Lie type, and suppose that

$$(G, i) \neq (E_6, 2), \quad (E_6, 4).$$

Then, for every self-map f of X with nonzero Kahler degree $deg(f) = m$,

$$f^* = \psi^m.$$

4. Two related questions

For a generalized Grassmannian X, taking Kahler degrees gives rise to a map

$$\deg : [X, X] \to \mathbb{Z}, \quad [f] \to \deg(f),$$

which defines a partition on the homotopy set $[X, X]$:

$$[X, X] = \amalg_{k \in \mathbb{Z}} [X, X]_k, \ [X, X]_k := \{[f] \in [X, X] \mid \deg(f) = k\}.$$

4. Two related questions

For a generalized Grassmannian X, taking Kahler degrees gives rise to a map

$$\deg : [X, X] \to \mathbb{Z}, \quad [f] \to \deg(f),$$

which defines a partition on the homotopy set $[X, X]$:

$$[X, X] = \amalg_{k \in \mathbb{Z}} [X, X]_k, \ [X, X]_k := \{[f] \in [X, X] \mid \deg(f) = k\}.$$

Theorem B provides no information about the degree zero component $[X, X]_0$.

4. Two related questions

For a generalized Grassmannian X, taking Kahler degrees gives rise to a map

$$\deg : [X, X] \to \mathbb{Z}, \quad [f] \to \deg(f),$$

which defines a partition on the homotopy set $[X, X]$:

$$[X, X] = \amalg_{k \in \mathbb{Z}} [X, X]_k, \ [X, X]_k := \{[f] \in [X, X] \mid \deg(f) = k\}.$$

Theorem B provides no information about the degree zero component $[X, X]_0$.

Examples Previous information on $[X, X]_0$:
(1) For $X = G_{n,k}$, Hoffman (1984) conjectured that $[X, X]_0 = \{0\}$;

4. Two related questions

For a generalized Grassmannian X, taking Kahler degrees gives rise to a map

$$\deg : [X, X] \to \mathbb{Z}, \quad [f] \to \deg(f),$$

which defines a partition on the homotopy set $[X, X]$:

$$[X, X] = \amalg_{k \in \mathbb{Z}} [X, X]_k, \ [X, X]_k := \{[f] \in [X, X] \mid \deg(f) = k\}.$$

Theorem B provides no information about the degree zero component $[X, X]_0$.

Examples Previous information on $[X, X]_0$:
(1) For $X = G_{n,k}$, Hoffman (1984) conjectured that $[X, X]_0 = \{0\}$;
(2) For $X = \mathbb{C}S_n$, Duan (2002) proved that $[X, X]_0 \neq \{0\}$;

4. Two related questions

For a generalized Grassmannian X, taking Kahler degrees gives rise to a map

$$\deg : [X, X] \to \mathbb{Z}, \quad [f] \to \deg(f),$$

which defines a partition on the homotopy set $[X, X]$:

$$[X, X] = \amalg_{k \in \mathbb{Z}} [X, X]_k, \ [X, X]_k := \{[f] \in [X, X] \mid \deg(f) = k\}.$$

Theorem B provides no information about the degree zero component $[X, X]_0$.

Examples Previous information on $[X, X]_0$:
(1) For $X = G_{n,k}$, Hoffman (1984) conjectured that $[X, X]_0 = \{0\}$;
(2) For $X = \mathbb{C}S_n$, Duan (2002) proved that $[X, X]_0 \neq \{0\}$;
(3) For the eight generalized exceptional Grassmannians

$$X = \frac{F_4}{C_3 \cdot S^1}, \ \frac{F_4}{B_3 \cdot S^1}, \ \frac{E_6}{A_6 \cdot S^1}, \ \frac{E_6}{D_5 \cdot S^1}, \ \frac{E_7}{D_6 \cdot S^1}, \ \frac{E_7}{E_6 \cdot S^1}, \ \frac{E_9}{E_7 \cdot S^1},$$

Zhao proved that $[X, X]_0 = \{0\}$.

4. Two related questions

Question 1. Let X be generalized exceptional Grassmannian. Is there a general result about

$$[X, X]_0 = ?$$

that unifies above examples?

[9] X.Z. Lin, Geometric realization of Adams maps, Acta Math.Sin., Engl. Ser. 27, 2011.

4. Two related questions

Question 1. Let X be generalized exceptional Grassmannian. Is there a general result about

$$[X, X]_0 = ?$$

that unifies above examples?

The second question is

Question 2 (Realization of Adams operators) For which nonzero integer m, there exists a self-map $f : X \to X$ such that

$$\deg(f) = m \Leftrightarrow f^* = \psi^m ?$$

[9] X.Z. Lin, Geometric realization of Adams maps, Acta Math.Sin., Engl. Ser. 27, 2011.

4. Two related questions

Question 1. Let X be generalized exceptional Grassmannian. Is there a general result about

$$[X, X]_0 = ?$$

that unifies above examples?

The second question is

Question 2 (Realization of Adams operators) For which nonzero integer m, there exists a self-map $f : X \to X$ such that

$$\deg(f) = m \Leftrightarrow f^* = \psi^m ?$$

Concerning this question, Xianzu Lin [9] has the following partial answer.

[9] X.Z. Lin, Geometric realization of Adams maps, Acta Math.Sin., Engl. Ser. 27, 2011.

4. Two related questions

Question 1. Let X be generalized exceptional Grassmannian. Is there a general result about

$$[X, X]_0 = ?$$

that unifies above examples?

The second question is

Question 2 (Realization of Adams operators) For which nonzero integer m, there exists a self-map $f : X \to X$ such that

$$\deg(f) = m \Leftrightarrow f^* = \psi^m ?$$

Concerning this question, Xianzu Lin [9] has the following partial answer.

Proposition. Let $X = G/P$ be a flag manifold, and let m be an integer coprime to the order of the Weyl group W_G. Then there exists a self-map f of X such that

$$f^* = \psi^m.$$

[9] X.Z. Lin, Geometric realization of Adams maps, Acta Math.Sin., Engl. Ser. 27, 2011.

4. Two related questions

Remark: The content of this talk is available at

H.Duan, R.Huang, X.Zhao, A Classification of Self-Maps of Generalized Grassmannians, arxiv.org/abs/2608.25355.

H.Duan, Rational homotopy theory of flag manifolds, arxiv.org/abs/2609.30591.