Homotopy classification of S^{2k-1}-bundles over S^{2k}

Zhongjian Zhu

Wenzhou University

Joint with Prof. Jianzhong Pan

Outline

- Backgrounds and motivation
- Our results
- Sketch of proofs

This talk is based on the following work
1. Z.J. Zhu, J.Z. Pan, Homotopy classification of S^{2k-1}-bundles over S^{2k}, arXiv:2508.14341 (2025)
2. Z.J. Zhu, J.Z. Pan, Homotopy types of S^{2k-1}-bundles over S^{2k}, arXiv:2508.13800 (2025)

Backgrounds and motivation

• It is well known that the bundle equivalent classes of S^n-bundles over S^m with structure group SO_{n+1} (special orthogonal group) is classified by the homotopy groups $\pi_{m-1}(SO_{n+1})$

• If the bundle is replaced by a (Serre) fibration, a similar correspondence holds, except that SO_{n+1} is replaced by the space $G_{n+1} := \{f : S^n \to S^n \mid f \text{ is homotopy equivalent}\}$ with compact-open topology. [Stasheff, 1963[1]]

[1] J. Stasheff, *A classification theorem for fibre spaces*. Topology 2 (1963), 239–246.

Backgrounds and motivation

• It is well known that the bundle equivalent classes of S^n-bundles over S^m with structure group SO_{n+1} (special orthogonal group) is classified by the homotopy groups $\pi_{m-1}(SO_{n+1})$

• If the bundle is replaced by a (Serre) fibration, a similar correspondence holds, except that SO_{n+1} is replaced by the space $G_{n+1} := \{f : S^n \to S^n \mid f \text{ is homotopy equivalent}\}$ with compact-open topology. [Stasheff, 1963[1]]

[1] J. Stasheff, *A classification theorem for fibre spaces*. Topology 2 (1963), 239–246.

Backgrounds and motivation

• At first glance, such a classification problem for equivalent classes receives a perfect homotopy-theoretic solution.

• However, some other related problems are not thereby resolved. For example, the classification problem for the total spaces of these fiber bundles or Serre fibrations. Considering that non-equivalent fiber bundles can have homeomorphic (homotopic) total spaces, the classification of total spaces is non-trivial.

• James and Whitehead initiated such research for sphere bundles over spheres [2],[3] .

[2] I.M. James, J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres I*, Proc. London Math. Soc. 4 (1954), 196-218.

[3] I.M. James, J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres II*, Proc. London Math. Soc. 5 (1955), 148-166.

Backgrounds and motivation

If we consider S^{2k-1}-bundles (fibrations) over S^{2k}, their total spaces are CW-complexes with three cells:

$$X = (S^{2k-1} \cup_\phi e^{2k}) \cup e^{4k-1}.$$

This led them to study the classification problem for such CW-complexes. [Sasao, Topology, 1965] provided the homotopy classification of X in the special cases $k = 2, 4$.

Backgrounds and motivation

Sphere bundles over spheres are of great significance in geometric topology.

• Milnor showed that the total spaces of S^3-bundles over S^4 with Euler class ± 1 are manifolds homeomorphic to S^7 but not always diffeomorphic to it.

• The unit tangent sphere bundle is diffeomorphic to the Stiefel manifold, i.e.,

$$STS^{2k} \cong V_{2k+1,2}.$$

• Grove and Ziller showed that the total space of any S^3-bundles over S^4 admits a metric with non-negative sectional curvature [4].

[4] K. Grove and W. Ziller, *Curvature and symmetry of Milnor spheres*, Ann. of Math. 152 (2000), 331–367

Backgrounds and motivation

• On the other hand, in homotopy theory, Mimura and Toda used $V_{2k+1,2}$ to construct a series of S^{2k-1}-bundles over spheres, whose total spaces become factors in the p-local product decomposition of the so-called "quasi-regular" compact Lie group [5].

• S.D. Theriault constructed a special S^{2k-1}-fibration over S^{2k} to make progress on computing the homotopy exponents of mod 2^r Moore spaces [6].

---

[5] M. Mimura, H. Toda, *Cohomology Operations and Homotopy of Compact Lie Groups I*, Topology 9 (1970), 317–336

[6] S.D. Theriault, *Homotopy exponents of mod 2^r Moore spaces*, Topology 6(2008) 369-398

Backgrounds and motivation

• In these works, understanding the total space of the sphere bundle (fibrations) is crucial.

• Note that the mapping ϕ in the aforementioned cellular CW-complex $X = (S^{2k-1} \cup_\phi e^{2k}) \cup e^{4k-1}$ is easily computed. Therefore, determining the total space of such sphere bundles essentially reduces to the attaching map of the top-dimensional cell. Hence, determining the total space of such sphere bundles can be divided into the following two problems:

(i) Determine the necessary and sufficient conditions for $f : S^{4k-2} \to P^{2k}(n)$ to be the attaching map of the top-dimensional cell in an S^{2k-1}-bundle (fibration) over S^{2k}, where $P^{2k}(n) = S^{2k-1} \cup_\phi e^{2k}$;

(ii) Determine the homotopy equivalence classes of the mapping cones C_f of f such that C_f is a homotopy type of total space of S^{2k-1}-bundle (fibration) over S^{2k}.

Idea:

• $\alpha \neq \beta \in \pi_{m-1}(SO_{n+1}) \iff \mathit{Bundle}(\alpha) \ncong \mathit{Bundle}(\beta)$
$\Rightarrow$ The total spaces of $\mathit{Bundle}(\alpha)$ and $\mathit{Bundle}(\beta)$ may be homotopic to each other.

• Find conditions on $\alpha, \beta \in \pi_{m-1}(SO_{n+1})$ such that the total spaces of $\mathit{Bundle}(\alpha)$ and $\mathit{Bundle}(\beta)$ are homotopic to each other.

• [I.M.James, J.H.C. Whitehead. Proc. LMS. 1954-1955][7],[8] give some sufficient and necessary conditions for totol spaces of S^r-bundles over S^q to have the same homotopy type, and classify the the total spaces of S^r-bundles over S^q for $q \leq 6$ (excluding S^3-bundles over S^4 without sections) up to homotopy equivalence;

• [I. Tamur, J. Math. Soc. Jpn. 1957] give a sufficient condition for total spaces of S^r-bundles over S^q to have the same homotopy type, when $(r, q) = (3, 4), (7, 8), (r \geq 4, q = 4), (r \geq 8, q = 8)$;

• [D. Crowley, C. M. Escher, Differ. Geom. Appl. 2003] classify the total spaces of S^3-bundles over S^4 up to homotopy equivalence,a homeomorphism and diffeomorphism.

• $\cdots$

---

[7] I.M. James, J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres I*, Proc. London Math. Soc. 4 (1954), 196-218.

[8] I.M. James, J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres II*, Proc. London Math. Soc. 5 (1955), 148-166.

Problem: What conditions must f satisfy for the space $X = P^{2k}(n) \cup_f e^{4k-1}$ to be the homotopy type of total space of S^{2k-1}-bundle over S^{2k}? $f \in \pi_{4k-2}(P^{2k}(n))$.
[Kitchloo-Shankar, IMRN, 2001] [9] consider the case $k = 2$, i.e., criterion (on $H^*(X)$) for determining whether a given CW-complex is homotopy equivalent to an S^3-fibration (bundle) over S^4.

Our research last year coincidentally focused on computing unstable homotopy groups of complexes:

- Zhu, Pan, The 2-local unstable homotopy groups of A_3^2-Complexes, Science China Mathematics, vol.67(3), 607-626,2024.3
- Zhu, Jin, The relative James construction and its application to homotopy groups, Topology and its Applications, vol. 356 , 109043, 2024.8
- Zhu, The unstable homotopy groups of 2-cell complexes, p.1-72 arxiv20416v2, 2024

[9] N. Kitchloo, S. Krishnan, *On complexes equivalent to S^3-bundles over S^4*, Int. Math. Res. Notices 8 (2001), 381-394.

Problem: What conditions must f satisfy for the space $X = P^{2k}(n) \cup_f e^{4k-1}$ to be the homotopy type of total space of S^{2k-1}-bundle over S^{2k}? $f \in \pi_{4k-2}(P^{2k}(n))$.

Problem: What conditions must f satisfy for the space $X = P^{2k}(n) \cup_f e^{4k-1}$ to be the homotopy type of total space of S^{2k-1}-fibration over S^{2k}? $f \in \pi_{4k-2}(P^{2k}(n))$.

Problem: What conditions must f satisfy for the space $X = P^{2k}(n) \cup_f e^{4k-1}$ to be the homotopy type of total space of S^{2k-1}-bundle over S^{2k}? $f \in \pi_{4k-2}(P^{2k}(n))$.

Problem: What conditions must f satisfy for the space $X = P^{2k}(n) \cup_f e^{4k-1}$ to be the homotopy type of total space of S^{2k-1}-fibration over S^{2k}? $f \in \pi_{4k-2}(P^{2k}(n))$.

Problem: What conditions must f satisfy for the space $X = P^{2k}(n) \cup_f e^{4k-1}$ to be the homotopy type of total space of S^{2k-1}-fibration over S^{2k}? $f \in \pi_{4k-2}(P^{2k}(n))$.

$$\pi_{4k-2}(P^{2k}(n)) \xrightarrow{i_*} \pi_{4k-2}(P^{2k}(n), S^{2k-1}) \xrightarrow{\partial} \pi_{4k-3}(S^{2k-1}).$$

Why put f into $\pi_{4k-2}(P^{2k}(n), S^{2k-1})$ by i_*?

James' Theorem 1957 [10]: for $X = (S^q \cup_\alpha e^t) \cup_f e^{t+q}$, $t \geq q+2$, $i_*(f)$ reflects the **cup product** on $H^*(X)$ where $\pi_{t+q-1}(S^q \cup_\alpha e^t) \xrightarrow{i_*} \pi_{t+q-1}(S^q \cup_\alpha e^t, S^{2k-1})$;

" $X \simeq$ a total space of bundle" $\Rightarrow$ " X satisfies the Poincare Duality" $\Leftrightarrow$ conditions on **cup product** on $H^*(X)$

[10] I.M. James, *Note on cup-products*, Proc. Amer. Math. Soc. 8 (1957), 374-383.

Problem: What conditions must f satisfy for the space $X = P^{2k}(n) \cup_f e^{4k-1}$ to be the homotopy type of total space of S^{2k-1}-fibration over S^{2k}? $f \in \pi_{4k-2}(P^{2k}(n))$.

$$\pi_{4k-2}(P^{2k}(n)) \xrightarrow{i_*} \pi_{4k-2}(P^{2k}(n), S^{2k-1}) \xrightarrow{\partial} \pi_{4k-3}(S^{2k-1}).$$

Why put f into $\pi_{4k-2}(P^{2k}(n), S^{2k-1})$ by i_*?

James' Theorem 1957 [10]: for $X = (S^q \cup_\alpha e^t) \cup_f e^{t+q}$, $t \geq q+2$, $i_*(f)$ reflects the **cup product** on $H^*(X)$ where $\pi_{t+q-1}(S^q \cup_\alpha e^t) \xrightarrow{i_*} \pi_{t+q-1}(S^q \cup_\alpha e^t, S^{2k-1})$;

" $X \simeq$ a total space of bundle" $\Rightarrow$ " X satisfies the Poincare Duality" $\Leftrightarrow$ conditions on **cup product** on $H^*(X)$

[10] I.M. James, *Note on cup-products*, Proc. Amer. Math. Soc. 8 (1957), 374-383.

Problem: What conditions must f satisfy for the space $X = P^{2k}(n) \cup_f e^{4k-1}$ to be the homotopy type of total space of S^{2k-1}-fibration over S^{2k}? $f \in \pi_{4k-2}(P^{2k}(n))$.

$$\pi_{4k-2}(P^{2k}(n)) \xrightarrow{i_*} \pi_{4k-2}(P^{2k}(n), S^{2k-1}) \xrightarrow{\partial} \pi_{4k-3}(S^{2k-1}).$$

Why put f into $\pi_{4k-2}(P^{2k}(n), S^{2k-1})$ by i_*?

James' Theorem 1957 [10]: for $X = (S^q \cup_\alpha e^t) \cup_f e^{t+q}$, $t \geq q+2$, $i_*(f)$ reflects the **cup product** on $H^*(X)$ where $\pi_{t+q-1}(S^q \cup_\alpha e^t) \xrightarrow{i_*} \pi_{t+q-1}(S^q \cup_\alpha e^t, S^{2k-1})$;

" $X \simeq$ a total space of bundle" $\Rightarrow$ " X satisfies the Poincare Duality" $\Leftrightarrow$ conditions on **cup product** on $H^*(X)$

[10] I.M. James, *Note on cup-products*, Proc. Amer. Math. Soc. 8 (1957), 374-383.

Problem: What conditions must f satisfy for the space $X = P^{2k}(n) \cup_f e^{4k-1}$ to be the homotopy type of total space of S^{2k-1}-fibration over S^{2k}? $\quad f \in \pi_{4k-2}(P^{2k}(n))$.

$$\pi_{4k-2}(P^{2k}(n)) \xrightarrow{i_*} \pi_{4k-2}(P^{2k}(n), S^{2k-1}) \xrightarrow{\partial} \pi_{4k-3}(S^{2k-1}).$$

Results for $\pi_{4k-2}(P^{2k}(n), S^{2k-1})$ [11]

$\pi_{2k}(P^{2k}(n), S^{2k-1}) = \mathbb{Z}\{X_{2k}\}$; $\pi_{2k-1}(S^{2k-1}) = \mathbb{Z}\{\iota_{2k-1}\}$, where X_{2k} is the the characteristic map of $2k$-cell in $P^{2k}(n)$ and ι_{2k-1} is identity map of S^{2k-1}. Let $[X_{2k}, \iota_{2k-1}]_r \in \pi_{4k-2}(P^{2k}(n), S^{2k-1})$ be the **relative Whitehead product** of the generators X_{2k} and ι_{2k-1}.

There is the exact sequence

$$0 \to \{[X_{2k}, \iota_{2k-1}]_r\} \xrightarrow{i} \pi_{4k-2}(P^{2k}(n), S^{2k-1}) \xrightarrow{\bar{\pi}_*} \pi_{4k-2}(S^{2k}) \to 0.$$

$$\pi_{4k-2}(P^{2k}(n), S^{2k-1}) = \begin{cases} \mathbb{Z}_n\{[X_{2k}, \iota_{2k-1}]_r\} \oplus \pi_{4k-2}(S^{2k}), & k = 2, 4; \\ \mathbb{Z}_{2n}\{[X_{2k}, \iota_{2k-1}]_r\} + X_{2k*}\pi_{4k-2}(D^{2k}, S^{2k-1}), & k \neq 2, 4. \end{cases}$$

[11] S. Sasao, On homotopy groups $\pi_{2n}(K^n_m, S^n)$, Proc. Jap. Acad. 39 (1963)

Our results

$$\pi_{4k-2}(P^{2k}(n)) \xrightarrow{i_*} \pi_{4k-2}(P^{2k}(n), S^{2k-1}) \xrightarrow{\partial} \pi_{4k-3}(S^{2k-1}).$$

There is the exact sequence

$$0 \to \{[X_{2k}, \iota_{2k-1}]_r\} \xrightarrow{i} \pi_{4k-2}(P^{2k}(n), S^{2k-1}) \xrightarrow{\bar{\pi}_*} \pi_{4k-2}(S^{2k}) \to 0.$$

$$\pi_{4k-2}(P^{2k}(n), S^{2k-1}) = \begin{cases} \mathbb{Z}_n\{[X_{2k}, \iota_{2k-1}]_r\} \oplus \pi_{4k-2}(S^{2k}), & k = 2, 4; \\ \mathbb{Z}_{2n}\{[X_{2k}, \iota_{2k-1}]_r\} + X_{2k*}\pi_{4k-2}(D^{2k}, S^{2k-1}), & k \neq 2, 4. \end{cases}$$

Theorem 1 (Zhu-Pan, preprint I 2025[12])

Let $X = P^{2k}(n) \cup_f e^{4k-1}$ be a simply connected CW complex. Then X is homotopy equivalent to an S^{2k-1}-fibration over S^{2k} if and only if $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n}$ with $\tau \in \mathbb{Z}_n^*$, where $\mathbb{Z}_n^*$ denotes the group of all invertible elements (in the sense of multiple operation) in $\mathbb{Z}_n$.

[12] Z.J. Zhu, J.Z. Pan, *Homotopy classification of S^{2k-1}-bundles over S^{2k}*, Preprint.

For $k = 2$, $X = P^4(n) \cup_f e^7$, $[X]$ is a specified generator of $H_7(X) \cong \mathbb{Z}$. We define the linking form as $b : H^4(X) \otimes H^4(X) \to \mathbb{Z}_n$, $x \otimes y \mapsto \langle \beta^{-1}(x), y \cap [X] \rangle$, where $\beta : H^3(X; \mathbb{Z}_n) \to H^4(X)$ is the Bockstein isomorphism and $\langle -, - \rangle : H^3(X; \mathbb{Z}_n) \otimes H_3(X) \to \mathbb{Z}_n$ is the Kronecker pairing.

Thm [Kitchloo-Shankar, IMRN, 2001] [13]

Let X be a simply connected CW-complex as above. Then X is homotopy equivalent to an S^3 -bundle over S^4 if and only if the following two conditions hold.

(I) The secondary cohomology operation Θ is trivial, where $\Theta : H^4(X; \mathbb{Z}_2) \to H^7(X; \mathbb{Z}_2)$ corresponds to the relation $Sq^2 Sq^2 = Sq^3 Sq^1$ in the mod 2 Steenrod algebra. [14]

(II) The linking form $b : H^4(X) \otimes H^4(X) \to \mathbb{Z}_n$ is equivalent to a standard form for some choice of $[X]$; that is, there exists an isomorphism $\psi : \mathbb{Z}_n \to H^4(X)$ such that $b(\psi(x), \psi(y)) = xy$.

[13] N. Kitchloo, S. Krishnan, *On complexes equivalent to S^3-bundles over S^4*, Int. Math. Res. Notices 8 (2001), 381-394.

[14] Harper, Secondary cohomology operations, AMS, Providence, RI, 2002

For $k = 2$, $X = P^4(n) \cup_f e^7$, $[X]$ is a specified generator of $H_7(X) \cong \mathbb{Z}$. We define the linking form as $b : H^4(X) \otimes H^4(X) \to \mathbb{Z}_n$, $x \otimes y \mapsto \langle \beta^{-1}(x), y \cap [X] \rangle$, where $\beta : H^3(X; \mathbb{Z}_n) \to H^4(X)$ is the Bockstein isomorphism and $\langle -, - \rangle : H^3(X; \mathbb{Z}_n) \otimes H_3(X) \to \mathbb{Z}_n$ is the Kronecker pairing.

Thm [Kitchloo-Shankar, IMRN, 2001] [13]

Let X be a simply connected CW-complex as above. Then X is homotopy equivalent to an S^3 -bundle over S^4 if and only if the following two conditions hold.

(I) The secondary cohomology operation Θ is trivial, where $\Theta : H^4(X; \mathbb{Z}_2) \to H^7(X; \mathbb{Z}_2)$ corresponds to the relation $Sq^2 Sq^2 = Sq^3 Sq^1$ in the mod 2 Steenrod algebra. [14]

(II) The linking form $b : H^4(X) \otimes H^4(X) \to \mathbb{Z}_n$ is equivalent to a standard form for some choice of $[X]$; that is, there exists an isomorphism $\psi : \mathbb{Z}_n \to H^4(X)$ such that $b(\psi(x), \psi(y)) = xy$.

---

[13] N. Kitchloo, S. Krishnan, *On complexes equivalent to S^3-bundles over S^4*, Int. Math. Res. Notices 8 (2001), 381-394.

[14] Harper, Secondary cohomology operations, AMS, Providence, RI, 2002

For $k=2$, $X = P^4(n) \cup_f e^7$, $[X]$ is a specified generator of $H_7(X) \cong \mathbb{Z}$. We define the linking form as $b : H^4(X) \otimes H^4(X) \to \mathbb{Z}_n$, $x \otimes y \mapsto \langle \beta^{-1}(x), y \cap [X] \rangle$, where $\beta : H^3(X; \mathbb{Z}_n) \to H^4(X)$ is the Bockstein isomorphism and $\langle -, - \rangle : H^3(X; \mathbb{Z}_n) \otimes H_3(X) \to \mathbb{Z}_n$ is the Kronecker pairing.

Thm [Kitchloo-Shankar, IMRN, 2001] [13]

Let X be a simply connected CW-complex as above. Then X is homotopy equivalent to an S^3 -bundle over S^4 if and only if the following two conditions hold.

(I) The secondary cohomology operation Θ is trivial, where $\Theta : H^4(X; \mathbb{Z}_2) \to H^7(X; \mathbb{Z}_2)$ corresponds to the relation $Sq^2 Sq^2 = Sq^3 Sq^1$ in the mod 2 Steenrod algebra. [14]

(II) The linking form $b : H^4(X) \otimes H^4(X) \to \mathbb{Z}_n$ is equivalent to a standard form for some choice of $[X]$; that is, there exists an isomorphism $\psi : \mathbb{Z}_n \to H^4(X)$ such that $b(\psi(x), \psi(y)) = xy$.

[13] N. Kitchloo, S. Krishnan, *On complexes equivalent to S^3-bundles over S^4*, Int. Math. Res. Notices 8 (2001), 381-394.

[14] Harper, Secondary cohomology operations, AMS, Providence, RI, 2002

Our results

Theorem 2

Let $X = P^{2k}(n) \cup_f e^{4k-1}$. Then X is homotopy equivalent to an S^{2k-1}-fibration over S^{2k} if and only if the following two conditions hold:

(I) The induced map $\pi^{2k}(X) \xrightarrow{i_X^*} \pi^{2k}(P^{2k}(n)) \cong \mathbb{Z}_n$ is a surjection, where $\pi^{2k}(-) = [-, S^{2k}]$ is the cohomotopy group functor.

(II) The linking form $b : H^{2k}(X) \otimes H^{2k}(X) \to \mathbb{Z}_n$ is equivalent to a standard form for some choice of orientation on X i.e. there exists an isomorphism $\psi : \mathbb{Z}_n \to H^{2k}(X)$ such that $b(\psi(x), \psi(y)) = xy$.

Remark 3

By the exact sequence

$[S^{4k-2}, S^{2k}] \xleftarrow{f^*} [P^{2k}(n), S^{2k}] = \mathbb{Z}_n\{p_{2k}\} \xleftarrow{i_X^*} [X, S^{2k}]$, The condition (I) is equivalent to $p_{2k*}(f) = p_{2k} \circ f = 0$. For $k = 2$, the condition (I) of Theorem 2 is equivalent to the condition (I) of Thm [Kitchloo-Shankar, IMRN, 2001] .

By above theorem, if $X = P^{2k}(n) \cup_f e^{4k-1}$ is an S^{2k-1}-fibration over S^{2k}, then

$$f \in K_k^n = Ker(p_{2k*} : \pi_{4k-2}(P^{2k}(n)) \to \pi_{4k-2}(S^{2k}))$$

$$\pi_{4k-1}(P^{2k}(n), S^{2k-1}) \xrightarrow{\partial} \pi_{4k-2}(S^{2k-1}) \xrightarrow{i_{2k-1*}} \pi_{4k-2}(P^{2k}(n)) \xrightarrow{i_*} \pi_{4k-2}(P^{2k}(n), S^{2k-1})$$

$\partial\pi_{4k-1}(P^{2k}(n), S^{2k-1}) = n\pi_{4k-2}(S^{2k-1})$ [Sasao, 1965] , there is a short exact sequence

$$0 \to \pi_{4k-2}(S^{2k-1})/n\pi_{4k-2}(S^{2k-1}) \overset{i_{2k-1*}}{\to} K_k^n \overset{i_*}{\to} i_*(K_k^n) \to 0 \quad (1)$$

Lemma 4

$$i_*(K_k^n) = \begin{cases} \mathbb{Z}_n\{[X_{2k}, \iota_{2k-1}]_r\}, & k = 2, 4; \\ \mathbb{Z}_{2n}\{[X_{2k}, \iota_{2k-1}]_r\}, & k \neq 2, 4 \text{ and } 2|n; \\ \mathbb{Z}_n\{2[X_{2k}, \iota_{2k-1}]_r\}, & k \neq 2, 4 \text{ and } 2 \nmid n. \end{cases} \quad (2)$$

$$f \in K_k^n = Ker(p_{2k*} : \pi_{4k-2}(P^{2k}(n)) \to \pi_{4k-2}(S^{2k}))$$

$$i_*(K_k^n) = \begin{cases} \langle [X_{2k}, \iota_{2k-1}]_r \rangle, & k = 2, 4 \text{ or } 2|n; \\ \langle 2[X_{2k}, \iota_{2k-1}]_r \rangle, & k \neq 2, 4 \text{ and } 2 \nmid n. \end{cases}$$

$$0 \to \pi_{4k-2}(S^{2k-1})/n\pi_{4k-2}(S^{2k-1}) \xrightarrow{i_{2k-1*}} K_k^n \xrightarrow{i_*} i_*(K_k^n) \to 0$$

$$\theta_k^n \longmapsto \text{“2”}[X_{2k}, \iota_{2k-1}]_r$$

Let $\theta_k^n \in \pi_{4k-2}(P^{2k}(n))$ be the lift of $[X_{2k}, \iota_{2k-1}]_r$ for $k = 2, 4$ or $2|n$ (resp. $2[X_{2k}, \iota_{2k-1}]_r$ for $k \neq 2, 4$ and $2 \nmid n$) by the map i_*.

$$K_k^n = \langle \theta_k^n \rangle + i_{2k-1*}\pi_{4k-2}(S^{2k-1})/ni_{2k-1*}\pi_{4k-2}(S^{2k-1}). \quad (3)$$

where $\langle \theta_k^n \rangle$ denotes the cyclic subgroup of K_k^n generated by θ_k^n.

Theorem 5

Let $X = P^{2k}(n) \cup_f e^{4k-1}$, then X is homotopy equivalent to an S^{2k-1}-fibration over S^{2k} if and only if $f = a\theta_k^n + i_{2k-1} \circ \gamma$ where $\gamma \in \pi_{4k-2}(S^{2k-1})$ and there is a $\tau \in \mathbb{Z}_n^*$ such that

$$\begin{cases} 2a \equiv \pm\tau^2 \ (mod\ n), & k \neq 2, 4, 2 \nmid n; \\ a \equiv \pm\tau^2 \ (mod\ n), & otherwise. \end{cases}$$

Theorem 5 Let $X = P^{2k}(n) \cup_f e^{4k-1}$, then X is homotopy equivalent to an S^{2k-1}-fibration over S^{2k} if and only if $f = a\theta_k^n + i_{2k-1} \circ \gamma$ where $\gamma \in \pi_{4k-2}(S^{2k-1})$ and there is a $\tau \in \mathbb{Z}_n^*$ such that $\begin{cases} 2a \equiv \pm\tau^2 \pmod{n}, & k \neq 2, 4, 2 \nmid n; \\ a \equiv \pm\tau^2 \pmod{n}, & \text{otherwise.} \end{cases}$

- For special k, we compute $\pi_{4k-2}(P^{2k}(n))$, $K_k^n \subset \pi_{4k-2}(P^{2k}(n))$

Proposition 6 (Homotopy Equivalence[15])

$P^{2k}(n) \cup_{f_1} e^{4k-1} \simeq P^{2k}(n) \cup_{f_2} e^{4k-1}$ if and only if there is a homotopy equivalence $g \in \varepsilon(P^{2k}(n))$ such that $gf_1 \simeq \pm f_2$.

- These enable us to count the number G_k^n of the homotopy types of the total space of S^{2k-1}-fibration over S^{2k} for specific k.

[15] K. Yamaguchi. Self-homotopy equivalences and highly connected Poincare complexes, Lecture Notes in Mathematics 1425. (1990)

Theorem 7 (Zhu-Pan Preprint II 2025[16])

Let $n = 2^r p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ denote the prime factorization of n, where $r \geq 0$, $p_1 < p_2 < \cdots < p_s$ are odd primes, $e_1, e_2, \cdots, e_s > 0$. If $r = 0, 1$ and $p_i \equiv 1 \pmod 4$, $i = 1, \cdots, s$, then we say that n satisfies $\bigstar$. Let the greatest common divisor of two integers m and n be a. If n satisfies $\bigstar$, we denote it by $(m, n) = a^*$. Otherwise, by $(m, n) = a$. Then G_k^n for $2 \leq k \leq 6$ are given as follows

$(12, n) =$	2^*	$2^r t, r = 0, 1; t = 1, 3$
$G_2^n =$	1	$\frac{(r+1)(t+1)}{2}$

$(8, n) =$	2 or 4	otherwise
$G_3^n =$	1	2

$(240, n) =$	2^*	5^*	10^*	$2^r t_1 t_2, r = 0, 1, 2, 3, 4; t_1 = 1, 3; t_2 = 1, 5$
$G_4^n =$	1	2	3	$\frac{(r+1)(t_1+1)(t_2+1)}{4}$

$(8, n) =$	1	2^* or 8	2 or 4
$G_5^n =$	1	8	16

$(504, n) =$	2^*	$2^r t_1 t_2, r = 0, 1, 2, 3; t_1 = 1, 3, 9; t_2 = 1, 7$		
$G_6^n =$	2	$\frac{(t_1+1)(t_2+1)}{4}, r = 0$	$\frac{(r+1)(t_1+1)(t_2+1)}{2}, r = 1, 2$	$\frac{5(t_1+1)(t_2+1)}{4}, r = 3$

[16] Z.J. Zhu, J.Z. Pan, *Homotopy classification of S^{2k-1}-bundles over S^{2k}*, Preprint.

In order to compute the homotopy exponents of mod 2^r Moore spaces [17]. , S.D. Theriault constructed a special S^{2k-1}-fibration over S^{2k} and he called the total space "mod-2^r tangent bundles". This is a CW-complex $X = P^{2k}(n) \cup_f e^{4k-1}$ with $\Sigma^\infty f = 0$.

Theorem [Zhu-Pan 2025] For any $k \geq 2$, if $X = P^{2k}(n) \cup_f e^{4k-1}$ is homotopy equivalent to an S^{2k-1}-fibration over S^{2k} and $\Sigma^\infty \circ f = 0$, then the 2-local homotopy type of X is unique.

[17] S.D. Theriault, *Homotopy exponents of mod* 2^r *Moore spaces*, Topology 6(2008) 369-398

• We say two F-fibrations $X \xrightarrow{\pi} B$ and $X' \xrightarrow{\pi'} B$ are equivalent if there is an homotopy equivalence $f : X \to X'$, such that $f \circ \pi' = \pi$.

$$\begin{array}{ccccc} F & \longrightarrow & X & \xrightarrow{\pi} & B \\ & & \Big\downarrow f \simeq & & \Big\| \\ F & \longrightarrow & X' & \xrightarrow{\pi'} & B \end{array}.$$

• Let $G_{n+1} := \{f : S^n \to S^n \mid f \text{ is homotopy equivalent}\}$ with compact-open topology

• By Classification Theorem [Stasheff, 1963[18]], the classifying space BG_{n+1} for G_{n+1} exists and it classifies the equivalence classes of S^n-fibrations. Thus the (fibaration) equivalence classes of S^n-fibration over S^m is one-to-one correspondence with the $\pi_m(BG_{n+1})$.

• It is well known that the classifying space BO_{n+1} for orthogonal group O_{n+1} classifies the isomorphism classes of $n+1$-dimensional real vector bundles. Hence equivalent classes of S^n-bundles over S^m are in one-to-one correspondence with $\pi_m(BO_{n+1})$

[18] J. Stasheff, *A classification theorem for fibre spaces*. Topology 2 (1963), 239–246.

The classical J homomorphism identifies O_{n+1} with the sub-monoid of G_{n+1} of linear actions. It induces a map

$$J_* : \pi_m(BO_{n+1}) \to \pi_m(BG_{n+1}),$$

which corresponds to the map from the set of equivalent classes of S^n-bundles over S^m to the set of equivalence classes of S^n-fibration over S^m.
The square is from the universal fibration $M \to EM \to BM$ for $M = O_{n+1}, G_{n+1}$ (for $M = G_{n+1}$, this is a quasi-fibtration)

$$\begin{array}{ccc} \pi_m(BO_{n+1}) & \xrightarrow{J_*} & \pi_m(BG_{n+1}) \\ \downarrow \cong & & \downarrow \\ \pi_{m-1}(O_{n+1}) & \xrightarrow{J_*} & \pi_{m-1}(G_{n+1}) \end{array}$$

where the left map is isomorphic for $m \geq 2$ since EO_{n+1} is contractible and the right map is isomorphic for $m \geq 3$ since EG_{n+1} is aspherical.

Therefore, for $m \geq 3$, any S^n-fibration over S^m is equivalent to an S^n-bundle over S^m if $J_* : \pi_{m-1}(O_{n+1}) \to \pi_{m-1}(G_{n+1})$ is an epimorphism .

Proposition 8 (Zhu-Pan, preprint I 2025)

$J_* : \pi_{2k-1}(O_{2k}) \to \pi_{2k-1}(G_{2k})$ is an epimorphism for $k = 2, 3, 4$ and not an epimorphism for $k = 5, 6$.

Corollary 9 (Zhu-Pan, preprint I 2025)

Theorem 1, Theorem 2, Theorem 5 and Theorem 7 still hold when S^{2k-1}-fibration over S^{2k} is replaced with S^{2k-1}-bundle over S^{2k} for $k = 2, 3$ and 4.

Generalization of the James' Theorem

- Let $L := S^q \cup_\alpha e^t$, $t \geq q+2, q \geq 2$; $\quad X := L \cup_f e^{t+q}$. i.e., $X := S^q \cup_\alpha e^t \cup_f e^{t+q}$ with e_q, e_t and e_{t+q} the cohomology classes which are carried by cells S^q, e^t, e^{t+q}

$$f \in \pi_{t+q-1}(L) \xrightarrow{i_*} \pi_{t+q-1}(L, S^q) \xrightarrow{\partial} \pi_{t+q-2}(S^q).$$

- $\pi_{n+q-1}(L, S^q) = \mathbb{Z}\{[\sigma, \iota_q]_r\} \oplus Im\sigma_*$.
$\pi_t(L, S^q) = \mathbb{Z}\{\sigma\}$, $\sigma : (D^t, S^{t-1}) \to (L, S^q)$ attaching map;
$\pi_q(S^q) = \mathbb{Z}\{\iota_q\}$

Theorem 10 (James 1957[19])

Let $X = L \cup_f e^{t+q}$, with $f \in \pi_{t+q-1}(L)$. If $i_*(f) = m[\sigma, \iota_q]_r + \sigma \circ \rho$, then $e_q \cup e_t = \pm m e_{q+t}$.

- We want to generalize above theorem to $X = P^{2k}(n) \cup_f e^{4k-1}$.

[19] I.M. James, *Note on cup-products*, Proc. Amer. Math. Soc. 8 (1957), 374-383.

Generalization of the James' Theorem

$$\pi_{4k-2}(P^{2k}(n)) \xrightarrow{i_*} \pi_{4k-2}(P^{2k}(n), S^{2k-1}) \xrightarrow{\partial} \pi_{4k-3}(S^{2k-1}).$$

$$\pi_{4k-2}(P^{2k}(n), S^{2k-1}) = \begin{cases} \mathbb{Z}_n\{[X_{2k}, \iota_{2k-1}]_r\} \oplus \pi_{4k-2}(S^{2k}), & k = 2, 4; \\ \mathbb{Z}_{2n}\{[X_{2k}, \iota_{2k-1}]_r\} + X_{2k*}\pi_{4k-2}(D^{2k}, S^{2k-1}), & k \neq 2, 4. \end{cases}$$

For $X = P^{2k}(n) \cup_f e^{4k-1}$, let $\bar{e}^X_{2k-1}, \bar{e}^X_{2k}, \bar{e}^X_{4k-1}$ be the module n cohomology classes which are carried by cells of X.

Theorem 11 (General James' Thm, Zhu-Pan Preprint I 2025)

Let $X = P^{2k}(n) \cup_f e^{4k-1}$, where $f \in \pi_{4k-2}(P^{2k}(n))$ with $p_{2k*}(f) = 0$, i.e., $f \in K^n_k$. If $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, then $\bar{e}^X_{2k} \cup \bar{e}^X_{2k-1} = \pm m\bar{e}^X_{4k-1}$, where the latter m is the module n reduction of the front.

$i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$

The method of the proof:

$$\begin{array}{ccccc} S^{4k-2} & \xrightarrow{f} & P^{2k}(n) & \longrightarrow & X . \\ \| & & \downarrow{\scriptstyle j_1+j_2p_{2k}} & & \downarrow \\ S^{4k-2} & \xrightarrow{(j_1+j_2p_{2k})\circ f} & P^{2k}(n)\vee S^{2k} & \longrightarrow & V \end{array}$$

$X = P^{2k}(n) \cup_f e^{4k-1}$ $\qquad$ $\bar{e}^X_{2k} \cup \bar{e}^X_{2k-1} = \pm m \bar{e}^X_{4k-1}$

$V = (P^{2k}(n) \vee S^{2k}) \cup_{(j_1+j_2p_{2k})\circ f} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m \bar{e}_{4k-1}$

$(j_1 + j_2 p_{2k})\circ f = j_{1*}(f) - m[j_1 i_{2k-1}, j_2]$

$(P^{2k}(n) \vee S^{2k}) \cup_{j_{1*}(f)} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm 0 \bar{e}_{4k-1}$

$(P^{2k}(n) \vee S^{2k}) \cup_{m[j_1 i_{2k-1}, j_2]} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m \bar{e}_{4k-1}$

$(P^{2k}(n) \vee S^{2k}) \cup_{[j_1 i_{2k-1}, j_2]} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm \bar{e}_{4k-1}$

$(S^{2k-1} \vee S^{2k}) \cup_{[j_1, j_2]} e^{4k-1} \simeq S^{2k-1} \times S^{2k}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}^S_{2k-1} = \pm \bar{e}_{4k-1}$

$i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$

The method of the proof:
$$\begin{array}{ccccc} S^{4k-2} & \xrightarrow{f} & P^{2k}(n) & \longrightarrow & X \\ \| & & \downarrow{\scriptstyle j_1+j_2p_{2k}} & & \downarrow \\ S^{4k-2} & \xrightarrow{(j_1+j_2p_{2k})\circ f} & P^{2k}(n)\vee S^{2k} & \longrightarrow & V \end{array}.$$

$X = P^{2k}(n) \cup_f e^{4k-1}$ $\qquad$ $\bar{e}^X_{2k} \cup \bar{e}^X_{2k-1} = \pm m \bar{e}^X_{4k-1}$

$V = (P^{2k}(n) \vee S^{2k}) \cup_{(j_1+j_2p_{2k})\circ f} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m \bar{e}_{4k-1}$

$(j_1 + j_2p_{2k})\circ f = j_{1*}(f) - m[j_1 i_{2k-1}, j_2]$

$(P^{2k}(n) \vee S^{2k}) \cup_{j_{1*}(f)} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm 0 \bar{e}_{4k-1}$

$(P^{2k}(n) \vee S^{2k}) \cup_{m[j_1 i_{2k-1}, j_2]} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m \bar{e}_{4k-1}$

$(P^{2k}(n) \vee S^{2k}) \cup_{[j_1 i_{2k-1}, j_2]} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm \bar{e}_{4k-1}$

$(S^{2k-1} \vee S^{2k}) \cup_{[j_1, j_2]} e^{4k-1} \simeq S^{2k-1} \times S^{2k}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}^S_{2k-1} = \pm \bar{e}_{4k-1}$

$i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$

The method of the proof:

$$\begin{array}{ccccc}
S^{4k-2} & \xrightarrow{f} & P^{2k}(n) & \longrightarrow & X \, . \\
\Vert & & \downarrow{\scriptstyle j_1+j_2p_{2k}} & & \downarrow \\
S^{4k-2} & \xrightarrow{(j_1+j_2p_{2k})\circ f} & P^{2k}(n) \vee S^{2k} & \longrightarrow & V
\end{array}$$

$X = P^{2k}(n) \cup_f e^{4k-1}$ $\qquad \bar{e}^X_{2k} \cup \bar{e}^X_{2k-1} = \pm m \bar{e}^X_{4k-1}$

$V = (P^{2k}(n) \vee S^{2k}) \cup_{(j_1+j_2p_{2k})\circ f} e^{4k-1}$ $\qquad \bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m \bar{e}_{4k-1}$

$(j_1 + j_2p_{2k})\circ f = j_{1*}(f) - m[j_1 i_{2k-1}, j_2]$

$(P^{2k}(n) \vee S^{2k}) \cup_{j_{1*}(f)} e^{4k-1}$ $\qquad \bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm 0 \bar{e}_{4k-1}$

$(P^{2k}(n) \vee S^{2k}) \cup_{m[j_1 i_{2k-1}, j_2]} e^{4k-1}$ $\qquad \bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m \bar{e}_{4k-1}$

$(P^{2k}(n) \vee S^{2k}) \cup_{[j_1 i_{2k-1}, j_2]} e^{4k-1}$ $\qquad \bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm \bar{e}_{4k-1}$

$(S^{2k-1} \vee S^{2k}) \cup_{[j_1, j_2]} e^{4k-1} \simeq S^{2k-1} \times S^{2k}$ $\qquad \bar{e}^S_{2k} \cup \bar{e}^S_{2k-1} = \pm \bar{e}_{4k-1}$

$i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$

The method of the proof:
$$\begin{array}{ccccc} S^{4k-2} & \xrightarrow{f} & P^{2k}(n) & \longrightarrow & X \; . \\ \| & & \downarrow{\scriptstyle j_1+j_2p_{2k}} & & \downarrow \\ S^{4k-2} & \xrightarrow{(j_1+j_2p_{2k})\circ f} & P^{2k}(n)\vee S^{2k} & \longrightarrow & V \end{array}$$

$X = P^{2k}(n) \cup_f e^{4k-1}$ $\qquad$ $\bar{e}^X_{2k} \cup \bar{e}^X_{2k-1} = \pm m\bar{e}^X_{4k-1}$

$V = (P^{2k}(n) \vee S^{2k}) \cup_{(j_1+j_2p_{2k})\circ f} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m\bar{e}_{4k-1}$

$(j_1 + j_2p_{2k})\circ f = j_{1*}(f) - m[j_1 i_{2k-1}, j_2]$

$(P^{2k}(n) \vee S^{2k}) \cup_{j_{1*}(f)} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm 0\bar{e}_{4k-1}$

$(P^{2k}(n) \vee S^{2k}) \cup_{m[j_1 i_{2k-1}, j_2]} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m\bar{e}_{4k-1}$

$(P^{2k}(n) \vee S^{2k}) \cup_{[j_1 i_{2k-1}, j_2]} e^{4k-1}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm\bar{e}_{4k-1}$

$(S^{2k-1} \vee S^{2k}) \cup_{[j_1, j_2]} e^{4k-1} \simeq S^{2k-1} \times S^{2k}$ $\qquad$ $\bar{e}^S_{2k} \cup \bar{e}^S_{2k-1} = \pm\bar{e}_{4k-1}$

$i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$

The method of the proof:
$$\begin{array}{ccccc} S^{4k-2} & \xrightarrow{f} & P^{2k}(n) & \longrightarrow & X \\ \Vert & & \downarrow{\scriptstyle j_1+j_2p_{2k}} & & \downarrow \\ S^{4k-2} & \xrightarrow{(j_1+j_2p_{2k})\circ f} & P^{2k}(n) \vee S^{2k} & \longrightarrow & V \end{array} .$$

$$X = P^{2k}(n) \cup_f e^{4k-1} \qquad \bar{e}^X_{2k} \cup \bar{e}^X_{2k-1} = \pm m \bar{e}^X_{4k-1}$$

$$V = (P^{2k}(n) \vee S^{2k}) \cup_{(j_1+j_2p_{2k})\circ f} e^{4k-1} \qquad \bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m \bar{e}_{4k-1}$$

$$(j_1 + j_2 p_{2k})\circ f = j_{1*}(f) - m[j_1 i_{2k-1}, j_2]$$

$$(P^{2k}(n) \vee S^{2k}) \cup_{j_{1*}(f)} e^{4k-1} \qquad \bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm 0 \bar{e}_{4k-1}$$

$$(P^{2k}(n) \vee S^{2k}) \cup_{m[j_1 i_{2k-1}, j_2]} e^{4k-1} \qquad \bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm m \bar{e}_{4k-1}$$

$$(P^{2k}(n) \vee S^{2k}) \cup_{[j_1 i_{2k-1}, j_2]} e^{4k-1} \qquad \bar{e}^S_{2k} \cup \bar{e}_{2k-1} = \pm \bar{e}_{4k-1}$$

$$(S^{2k-1} \vee S^{2k}) \cup_{[j_1, j_2]} e^{4k-1} \simeq S^{2k-1} \times S^{2k} \qquad \bar{e}^S_{2k} \cup \bar{e}^S_{2k-1} = \pm \bar{e}_{4k-1}$$

For CW-complex X, $a \in H^p(X, \mathbb{Z}_n), b \in H^q(X, \mathbb{Z}_n)$, with $a \cup b = 0$, where $p, q > 2$. Suppose that $H^{p+q-1}(X, \mathbb{Z}_n) = 0$. Then a function $h : \pi_{p+q-1}(X) \to \mathbb{Z}_n$ is defined as follows. Let $\lambda \in \pi_{p+q-1}(X)$. Let $X^* = X \cup_\lambda e^{p+q}$. Let c denote the modulo-n reduction of the integral cohomology class carried by the top cell. There are unique elements $a' \in H^p(X^*, \mathbb{Z}_n), b' \in H^q(X^*, \mathbb{Z}_n)$, which map into a, b, respectively, under the injection. Since $a \cup b = 0$, there is a unique $m \in \mathbb{Z}_n$ such that $a' \cup b' = mc$. Define $h(\lambda) = m$. We prove:

Theorem 12

(Generalize the Theorem 4.1 of James' theorem) The function h is a homomorphism of groups.

The proof of the Theorem 1

$$\begin{array}{ccccc}
K_k^n & \xrightarrow{\subset} & \pi_{4k-2}(P^{2k}(n)) & \xrightarrow{p_{2k*}} & \pi_{4k-2}(S^{2k}) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \cong} \\
Ker(\bar{\pi}_*) & \xrightarrow{\subset} & \pi_{4k-2}(P^{2k}(n), S^{2k-1}) & \xrightarrow{\bar{\pi}_*} & \pi_{4k-2}(S^{2k}, *) \\
 & & \downarrow{\scriptstyle \partial} & & \\
 & & \pi_{4k-3}(S^{2k-1}) & &
\end{array}$$

- Calculate $i_*(K_k^n)$: $i_*(K_k^n) = Ker(\bar{\pi}_*) \cap Ker(\partial)$, which implies

$$i_*(K_k^n) = \begin{cases} \mathbb{Z}_n\{[X_{2k}, \iota_{2k-1}]_r\}, & k=2,4; \\ \mathbb{Z}_{2n}\{[X_{2k}, \iota_{2k-1}]_r\}, & k\neq 2,4,\ 2|n; \\ \mathbb{Z}_n\{2[X_{2k}, \iota_{2k-1}]_r\}, & k\neq 2,4,\ 2\nmid n. \end{cases} \quad \text{by } Ker(\bar{\pi}_*) = \begin{cases} \mathbb{Z}_n\{[X_{2k}, \iota_{2k-1}]_r\}, & k=2,4; \\ \mathbb{Z}_{2n}\{[X_{2k}, \iota_{2k-1}]_r\}, & k\neq 2,4. \end{cases}$$

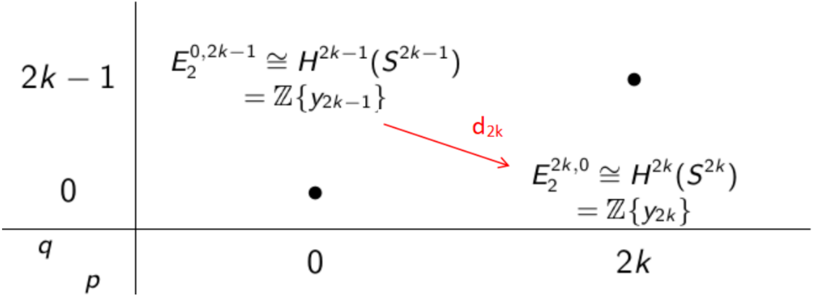
and $\partial([X_{2k}, \iota_{2k-1}]_r) = -[\partial X_{2k}, \iota_{2k-1}] = -n[\iota_{2k-1}, \iota_{2k-1}]$

- Prove $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$ for some m iff $p_{2k} \circ f = 0$ (i.e., $f \in K_k^n$) : $p_{2k} \circ f = 0 \Leftrightarrow \bar{\pi}_* i_* f = 0 \Leftrightarrow i_* f \in Ker(\bar{\pi}_*)$, i.e., $i_* f \in Ker(\bar{\pi}_*) \cap Ker(\partial) = i_*(K_k^n)$.

pf: $S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr. $\Rightarrow$ $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n}$

Note there is a cofib. seq. $S^{4k-2} \xrightarrow{f} P^{2k}(n) \xrightarrow{i_X} X$.

- Consider the Serre S.S. for the fibration converging to $H^*(X)$



(1) $E_2^{p,q} = E_{2k}^{p,q} = H^p(S^{2k}) \otimes H^q(S^{2k-1})$

(2) $d_{2k}(y_{2k-1}) = ny_{2k}$ where y_{2k-1} and y_{2k} are suitably chosen generators of $H^{2k-1}(S^{2k-1})$ and $H^{2k}(S^{2k})$, resp.

(3) $\pi^* : E_2^{2k,0} = H^{2k}(S^{2k}) \twoheadrightarrow E_\infty^{2k,0} \subset H^{2k}(X)$, and $E_\infty^{2k,0} = H^{2k}(X)$ which implies $\pi^* : H^{2k}(S^{2k}) \to H^{2k}(X)$ is surjective.

pf: $S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr. $\Rightarrow$ $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n}$

$\pi^* : H^{2k}(S^{2k}) \to H^{2k}(X)$ is surjective.

So is $(\pi \circ i_X)^*$ where $\pi \circ i_X : P^{2k}(n) \xrightarrow{i_X} X \xrightarrow{\pi} S^{2k}$.

Since $[P^{2k}(n), S^{2k}] = \mathbb{Z}_n\{p_{2k}\}$, Hence $\pi \circ i_X = a p_{2k}$, $a \in \mathbb{Z}_n^*$.

$\Rightarrow (a p_{2k}) \circ f = \pi \circ i_X \circ f = 0$ by cofib. seq. $S^{4k-2} \xrightarrow{f} P^{2k}(n) \xrightarrow{i_X} X$

$\Rightarrow p_{2k} \circ f = 0$. So $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$ for some integer m.

pf: $S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr. $\Rightarrow$ $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n}$

- For the Serre S.S. in $\mathbb{Z}_n$-coefficients converging to $H^*(X, \mathbb{Z}_n)$, we get $E_2^{p,q} = E_\infty^{p,q} = H^p(S^{2k}, \mathbb{Z}_n) \otimes H^q(S^{2k-1}, \mathbb{Z}_n)$.

$2k-1$	$E_2^{0,2k-1} \cong H^{2k-1}(S^{2k-1}; \mathbb{Z}_n) = \mathbb{Z}/n\{[y_{2k-1}]\}$	$\bullet$
0	$\bullet$	$E_2^{2k,0} \cong H^{2k}(S^{2k}; \mathbb{Z}_n) = \mathbb{Z}/n\{[y_{2k}]\}$
q / p	0	$2k$

where $[y_{2k-1}]$, the n-module of $y_{2k-1} \in H^{2k-1}(S^{2k-1}) = E_2^{0,2k-1}$,

- Since in both spectral sequences there are no extension problems in the filtration $0 \subset F_t^t \subset \cdots \subset F_0^t = H^t(X)$ or $H^t(X; \mathbb{Z}_n)$, hence we can identify $H^*(X)$ or $H^*(X, \mathbb{Z}_n)$ with the $(2k+1)$-th stage in the respective spectral sequences.
- For $y_{2k} \otimes y_{2k-1} \in H^{2k}(S^{2k}) \otimes H^{2k-1}(S^{2k-1}) = E_2^{2k,2k-1}$ can be regarded as generators of $H^{4k-1}(X)$.

pf: $S^{2k-1}\to X\xrightarrow{\pi} S^{2k}$ is a fibr. $\Rightarrow$ $i_*(f)=m[X_{2k},\iota_{2k-1}]_r$, $m\equiv\pm\tau^2 \pmod n$

$$H^{4k-1}(X)=\mathbb{Z}\{y_{2k}\otimes y_{2k-1}\}$$

• Choose $[X]\in H_{4k-1}(X)$ satisfies $\langle y_{2k}\otimes y_{2k-1},[X]\rangle=1$. Clearly, $[X]=\pm\hat{e}^X_{4k-1}$, where $\hat{e}^X_{4k-1}$ is the duality of $\bar{e}^X_{4k-1}$.

• By $d_{2k}(y_{2k-1})=ny_{2k}$, the Bockstein isomorphism $\beta: H^{2k-1}(X,\mathbb{Z}_n)\to H^{2k}(X)$ is given by $\beta([y_{2k-1}])=y_{2k}$.
Moreover, $\beta(\bar{e}^X_{2k-1})=\bar{e}^X_{2k}$.
If $\bar{e}^X_{2k-1}=\tau[y_{2k-1}]$ with $\tau\in\mathbb{Z}_n^*$, then $\bar{e}^X_{2k}=\tau[y_{2k}]$.

• By the Genaral James' Thm

$$\left\langle \bar{e}^X_{2k-1}\cup\bar{e}^X_{2k},[X]\right\rangle=\left\langle \pm m\bar{e}^X_{4k-1},\pm\hat{e}^X_{4k-1}\right\rangle=\pm m$$

On the other hand

$$\left\langle \bar{e}^X_{2k-1}\cup\bar{e}^X_{2k},[X]\right\rangle=\tau^2\left\langle [y_{2k-1}]\cup[y_{2k}],[X]\right\rangle\equiv\tau^2 \pmod n$$

So $m\equiv\pm\tau^2 \pmod n$.

pf: $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n}$ $\Rightarrow S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr.

- $p_{2k} \circ f = 0$. By cof. seq. $S^{4k-2} \xrightarrow{f} P^{2k}(n) \xrightarrow{i_X} X$, there is an extension $\tilde{\pi} : X \to S^{2k}$ of p_{2k}, i.e.,

$$\begin{array}{ccccc} S^{4k-2} & \xrightarrow{f} & P^{2k}(n) & \xrightarrow{i_X} & X \\ & & \downarrow{\scriptstyle p_{2k}} & \swarrow{\scriptstyle \tilde{\pi}} & \\ & & S^{2k} & & \end{array} .$$

We have fib. seq. $F_{\tilde{\pi}} \to X \to S^{2k}$.

Method: Show the only nontrivial reduced homology of $F_{\tilde{\pi}}$ is $\bar{H}_{2k-1}(F_{\tilde{\pi}}) = \mathbb{Z}$ by spectral seqs.

pf: $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n} \Rightarrow S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr.

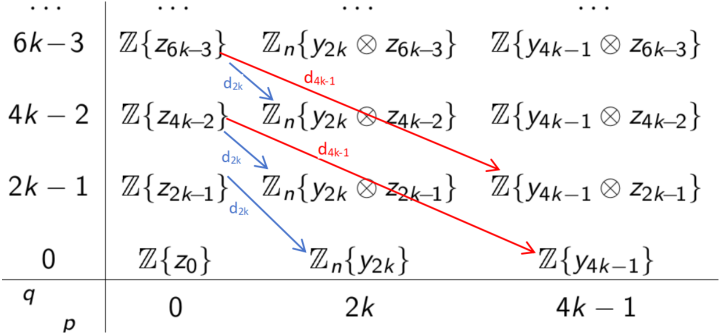
Consider Serre S.S. for the fib.: $\Omega S^{2k} \to F_{\tilde{\pi}} \to X$ and note that $H^*(\Omega S^{2k}) = \mathbb{Z}\{z_{(2k-1)j} \mid j = 0, 1, \cdots\}$.

$$E_2^{p,q} = H^p(X) \otimes H^q(\Omega S^{2k}) = \begin{cases} \mathbb{Z}\{z_{(2k-1)j}\}, & p = 0, q = (2k-1)j; \\ \mathbb{Z}_n\{y_{2k} \otimes z_{(2k-1)j}\}, & p = 2k, q = (2k-1)j; \\ \mathbb{Z}\{y_{4k-1} \otimes z_{(2k-1)j}\}, & p = 4k-1, q = (2k-1)j; \\ 0, & \text{otherwise.} \end{cases}$$

where y_{2k} and y_{4k-1} are generators of $H^{2k}(X)$ and $H^{4k-1}(X)$ resp. Simplify the generators $y_{2k} \otimes z_0$ and $y_{4k-1} \otimes z_0$ by y_{2k} and y_{4k-1} respectively in the following text.

pf: $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod n$ $\Rightarrow S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr.

The Serre S.S. for the fib.: $\Omega S^{2k} \to F_{\tilde{\pi}} \to X$ where $H^*(\Omega S^{2k}) = \mathbb{Z}\{z_{(2k-1)j} \mid j = 0, 1, \cdots\}$; y_{2k} and y_{4k-1} are generators of $H^{2k}(X)$. $E_2^{p,q} = H^p(X) \otimes H^q(\Omega S^{2k})$



Need to calculate the nontrivial differentials

$d_{2k} : E_{2k}^{0,(2k-1)j} = \mathbb{Z}\{z_{(2k-1)j}\} \to E_{2k}^{2k,(2k-1)(j-1)} = \mathbb{Z}_n\{y_{2k} \otimes z_{(2k-1)j}\}$;

$d_{4k-1} : E_{4k}^{0,(2k-1)j} = \mathbb{Z}\{nz_{(2k-1)j}\} \to E_{4k}^{4k-1,(2k-1)(j-2)} = \mathbb{Z}_n\{y_{4k-1} \otimes z_{(2k-1)j}\}$;

Calculate $d_{2k} : E_{2k}^{0,(2k-1)j} = \mathbb{Z}\{z_{(2k-1)j}\} \to E_{2k}^{2k,(2k-1)(j-1)} = \mathbb{Z}_n\{y_{2k} \otimes z_{(2k-1)j}\}$

Proposition 13

In the Serre S.S. for fib. $\Omega S^{2k} \to F_{\tilde{\pi}} \to X$, we have

$$d_{2k}(z_{(2k-1)j}) = y_{2k} \otimes z_{(2k-1)(j-1)}.$$

Proof.

It is not difficult to get the result by the naturality of the Serre spectral sequence with respect to maps of fibrations

$$\begin{array}{ccc} \Omega S^{2k} & \xrightarrow{=} & \Omega S^{2k} \\ \downarrow & & \downarrow \\ F_{\tilde{\pi}} & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tilde{\pi}} & S^{2k} \end{array}.$$

Calculate $d_{4k-1}: E_{4k}^{0,(2k-1)j} = \mathbb{Z}\{nz_{(2k-1)j}\} \to E_{4k}^{4k-1,(2k-1)(j-2)} = \mathbb{Z}_n\{y_{4k-1} \otimes z_{(2k-1)j}\}$

For an extension $\tilde{\pi}: X \to S^{2k}$ of p_{2k}, i.e. $i_X \circ \tilde{\pi} = p_{2k}$. Then there is the extension $f(\tilde{\pi})$ satisfies the homotopy commutative diagram

$$\begin{array}{ccccc} P^{2k}(n) & \xrightarrow{i_X} & X & \xrightarrow{p_X} & S^{4k-1} \\ \| & & \downarrow \tilde{\pi} & & \downarrow f(\tilde{\pi}) \\ P^{2k}(n) & \xrightarrow{p_{2k}} & S^{2k} & \xrightarrow{[n]} & S^{2k} \end{array}.$$

Proposition 14

The Hopf invariant, $H(f(\tilde{\pi}))$, is a multiple of n, i.e. $H(f(\tilde{\pi})) = \lambda n$, and in the Serre spectral sequence for the fibration $\Omega S^{2k} \to F_{\tilde{\pi}} \to X$, we have

$$d_{4k-1}(nz_{(2k-1)j}) = \lambda y_{4k-1} \otimes z_{(2k-1)(j-2)}, \quad j = 2, 3, \cdots.$$

Remark: Different choice of extensions $f(\tilde{\pi})$ of $\tilde{\pi}$ give the same Hopf invariant by $[X, S^{2k}] \xleftarrow{p_X^*} [S^{4k-1}, S^{2k}] \xleftarrow{(\Sigma f)^*} [P^{2k+1}(n), S^{2k}]$

Calculate $d_{4k-1} : E_{4k}^{0,(2k-1)j} = \mathbb{Z}\{nz_{(2k-1)j}\} \to E_{4k}^{4k-1,(2k-1)(j-2)} = \mathbb{Z}_n\{y_{4k-1} \otimes z_{(2k-1)j}\}$

$$\begin{array}{ccccc} P^{2k}(n) & \xrightarrow{i_X} & X & \xrightarrow{p_X} & S^{4k-1} \\ \| & & \downarrow \tilde{\pi} & & \downarrow f(\tilde{\pi}) \\ P^{2k}(n) & \xrightarrow{p_{2k}} & S^{2k} & \xrightarrow{[n]} & S^{2k} \end{array} . \qquad \begin{array}{ccc} X & \xrightarrow{\gamma} & S^{4k-1} \vee X \\ \downarrow \alpha\tilde{\pi} & \swarrow (\alpha,\tilde{\pi}) & \\ S^{2k} & & \end{array} .$$

From the exact sequence

$$[P^{2k}(n), S^{2k}] \xleftarrow{i_X^*} [X, S^{2k}] \xleftarrow{p_X^*} \pi_{4k-1}(S^{2k}) \xleftarrow{(\Sigma f)^*} [P^{2k+1}(n), S^{2k}],$$

there is an action $\pi_{4k-1}(S^{2k}) \curvearrowright [X, S^{2k}]$ defined by the left triangle above[Arkowitz, *Introduction to Homotopy Theory*] .

Lemma 15

If $\tilde{\pi} : X \to S^{2k}$ is an extension of p_{2k}, then any extension of p_{2k} is obtained by $\alpha\tilde{\pi}$ ($\alpha \in \pi_{4k-1}(S^{2k})$) which is defined by above right trangle where γ is the coaction induced by defining cofib. of X. Moreover, for the Hopf invariant $H : \pi_{4k-1}(S^{2k}) \to \mathbb{Z}$,

$$H(f(\alpha\tilde{\pi})) = n^2 H(\alpha) + H(f(\tilde{\pi})). \qquad (4)$$

Calculate $d_{4k-1} : E_{4k}^{0,(2k-1)j} = \mathbb{Z}\{nz_{(2k-1)j}\} \to E_{4k}^{4k-1,(2k-1)(j-2)} = \mathbb{Z}_n\{y_{4k-1} \otimes z_{(2k-1)j}\}$

Proof.

$$\begin{array}{ccccc} P^{2k}(n) & \xrightarrow{i_X} & X & \xrightarrow{p_X} & S^{4k-1} \\ \| & & \downarrow{\scriptstyle \tilde\pi} & & \downarrow{\scriptstyle f(\tilde\pi)} \\ P^{2k}(n) & \xrightarrow{p_{2k}} & S^{2k} & \xrightarrow{[n]} & S^{2k} \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\gamma} & S^{4k-1} \vee X \\ {\scriptstyle \alpha\tilde\pi}\downarrow & \swarrow{\scriptstyle (\alpha,\tilde\pi)} & \\ S^{2k} & & \end{array} .$$

• The "Moreover" part: note that $\alpha\tilde\pi$ fits into the following commutative diagrams:

$$\begin{array}{ccccc} X & \xrightarrow{\gamma} & S^{4k-1} \vee X & \xrightarrow{id \vee p_X} & S^{4k-1} \vee S^{4k-1} \\ {\scriptstyle \alpha\tilde\pi}\downarrow & & \downarrow{\scriptstyle (\alpha,\tilde\pi)} & & \downarrow{\scriptstyle ([n]\circ\alpha, f(\tilde\pi))} \\ S^{2k} & \xrightarrow[=]{} & S^{2k} & \xrightarrow{[n]} & S^{2k} \end{array} . \Rightarrow \begin{array}{ccc} X & \xrightarrow{p_X} & S^{4k-1} \\ {\scriptstyle \alpha\tilde\pi}\downarrow & & \downarrow{\scriptstyle [n]\circ\alpha + f(\tilde\pi)} \\ S^{2k} & \xrightarrow{[n]} & S^{2k} \end{array} .$$

$(id \vee p_X)\circ\gamma \simeq \nu\circ p_X$, where ν is the co-H-space structure map on S^{4k-1} and $([n]\circ\alpha, f(\tilde\pi))\circ\nu \simeq [n]\circ\alpha + f(\tilde\pi)$,
$H(f(\alpha\tilde\pi)) = H([n]\circ\alpha) + H(f(\tilde\pi)) = n^2 H(\alpha) + H(f(\tilde\pi))$. □

Prop 14. The Hopf invariant, $H(f(\tilde{\pi}))$, is a multiple of n, i.e. $H(f(\tilde{\pi})) = \lambda n$, and in the Serre S.S. for $\Omega S^{2k} \to F_{\tilde{\pi}} \to X$, we have

$$d_{4k-1}(nz_{(2k-1)j}) = \lambda y_{4k-1} \otimes z_{(2k-1)(j-2)}, \quad j = 2, 3, \cdots .$$

The Method of the proof

$$\begin{array}{ccc} F_{\tilde{\pi}} & \longrightarrow & G \\ \downarrow & & \downarrow \\ X & \xrightarrow{p_X} & S^{4k-1} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle f(\tilde{\pi})} \\ S^{2k} & \xrightarrow{[n]} & S^{2k} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \Omega S^{2k} & \xrightarrow{\Omega[n]} & \Omega S^{2k} \\ \downarrow & & \downarrow \\ F_{\tilde{\pi}} & \longrightarrow & G \\ \downarrow & & \downarrow \\ X & \xrightarrow{p_X} & S^{4k-1} \end{array} .$$

the Serre S.S. for $\Omega S^{2k} \to F_{\tilde{\pi}} \to X$

q \ p	0	$2k$	$4k-1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$6k-3$	$\mathbb{Z}\{z_{6k-3}\}$	$\mathbb{Z}_n\{y_{2k} \otimes z_{6k-3}\}$	$\mathbb{Z}\{y_{4k-1} \otimes z_{6k-3}\}$
$4k-2$	$\mathbb{Z}\{z_{4k-2}\}$	$\mathbb{Z}_n\{y_{2k} \otimes z_{4k-2}\}$	$\mathbb{Z}\{y_{4k-1} \otimes z_{4k-2}\}$
$2k-1$	$\mathbb{Z}\{z_{2k-1}\}$	$\mathbb{Z}_n\{y_{2k} \otimes z_{2k-1}\}$	$\mathbb{Z}\{y_{4k-1} \otimes z_{2k-1}\}$
0	$\mathbb{Z}\{z_0\}$	$\mathbb{Z}_n\{y_{2k}\}$	$\mathbb{Z}\{y_{4k-1}\}$

the Serre S.S. for $\Omega S^{2k} \to G \to S^{4k-1}$

q \ p	0	$4k-1$
$\cdots$	$\cdots$	$\cdots$
$4k-2$	$\mathbb{Z}\{z_{4k-2}\}$	$\mathbb{Z}\{\bar{y}_{4k-1} \otimes z_{4k-2}\}$
$2k-1$	$\mathbb{Z}\{z_{2k-1}\}$	$\mathbb{Z}\{\bar{y}_{4k-1} \otimes z_{2k-1}\}$
0	$\mathbb{Z}\{z_0\}$	$\mathbb{Z}\{\bar{y}_{4k-1}\}$

To prove: $d_{4k-1}(nz_{(2k-1)j}) = \lambda y_{4k-1} \otimes z_{(2k-1)(j-2)}$, for $j = 2$.

Calculate $d_{4k-1} : E_{4k}^{0,(2k-1)j} = \mathbb{Z}\{nz_{(2k-1)j}\} \to E_{4k}^{4k-1,(2k-1)(j-2)} = \mathbb{Z}_n\{y_{4k-1} \otimes z_{(2k-1)j}\}$

$H^*(\Omega S^{2k}) = \mathbb{Z}\{z_{(2k-1)j} \mid j = 0, 1, 2, \cdots\}$.

Proposition 16

Let $f : S^{4k-1} \to S^{2k}, k > 1$ be a map and G be the homotopy fiber of f. Then in the induced fibration sequence $\Omega S^{2k} \to G \to S^{4k-1}$, we have the identity $d_{4k-1}(z_{4k-2}) = H(f)\bar{y}_{4k-1}$, where $\bar{y}_{4k-1} \in H^{4k-1}(S^{4k-1})$ is a generator.

Proof:

$$\begin{array}{ccccc}
\pi_{4k-1}(S^{4k-1}) & \longleftarrow & \pi_{4k-1}(G, \Omega S^{2k}) & \xrightarrow{\partial} & \pi_{4k-2}(\Omega S^{2k}) \\
\downarrow h & & \downarrow h & & \downarrow h \\
H_{4k-1}(S^{4k-1}) & \longleftarrow & H_{4k-1}(G, \Omega S^{2k}) & \xrightarrow{\partial} & H_{4k-2}(\Omega S^{2k})
\end{array}$$

where the bottom row gives the transgression of the homology Serre spectral sequence associated to $\Omega S^{2k} \to G \to S^{4k-1}$.

$$\Rightarrow \qquad \begin{array}{ccc}
\pi_{4k-1}(S^{4k-1}) & \xrightarrow{\partial} & \pi_{4k-2}(\Omega S^{2k}) \\
\downarrow h & & \downarrow h \\
H_{4k-1}(S^{4k-1}) & \xrightarrow{d^{4k-1}} & H_{4k-2}(\Omega S^{2k})
\end{array}$$

Proof : $d_{4k-1}(z_{4k-2}) = H(f)\bar{y}_{4k-1}$, where $\bar{y}_{4k-1} \in H^{4k-1}(S^{4k-1})$

• $\pi_{4k-1}(S^{4k-1}) \xrightarrow{\partial} \pi_{4k-2}(\Omega S^{2k})$ is $\pi_{4k-2}(\Omega S^{4k-1}) \xrightarrow{(\Omega f)_*} \pi_{4k-2}(\Omega S^{2k})$.

$$\begin{array}{ccc} \pi_{4k-1}(S^{4k-1}) & \xrightarrow{\partial} & \pi_{4k-2}(\Omega S^{2k}) \\ \downarrow h\cong & & \downarrow h \\ H_{4k-1}(S^{4k-1}) & \xrightarrow{d^{4k-1}} & H_{4k-2}(\Omega S^{2k}) \end{array} \qquad \begin{array}{ccc} \pi_{4k-2}(\Omega S^{4k-1}) & \xrightarrow{(\Omega f)_*} & \pi_{4k-2}(\Omega S^{2k}) \\ \downarrow h\cong & & \downarrow h \\ H_{4k-2}(\Omega S^{4k-1}) & \xrightarrow{(\Omega f)_*} & H_{4k-2}(\Omega S^{2k}) \end{array}$$

• Prop 1.30 [20] implies $H_{4k-2}(\Omega S^{4k-1}) \xrightarrow{(\Omega f)_*} H_{4k-2}(\Omega S^{2k})$ sends a generator to $H(f)$ times a generator. The fact that related Hurewicz homomorphisms are all isomorphic implies that d^{4k-1} sends a generator to $H(f)$ times a generator.

• Dually, this gives the desired equality for differential in the cohomology Serre spectral sequence associated to $\Omega S^{2k} \to G \to S^{4k-1}$.

[20] A. Hatcher, *Spectral Sequences in Algebraic Topology*, (Unpublished) http://www.math. cornell.edu/ hatcher.

Calculate $d_{4k-1} : E_{4k}^{0,(2k-1)j} = \mathbb{Z}\{nz_{(2k-1)j}\} \to E_{4k}^{4k-1,(2k-1)(j-2)} = \mathbb{Z}_n\{y_{4k-1} \otimes z_{(2k-1)j}\}$

Prop 14. The Hopf invariant, $H(f(\tilde{\pi}))$, is a multiple of n, i.e. $H(f(\tilde{\pi})) = \lambda n$, and in the Serre S.S. for $\Omega S^{2k} \to F_{\tilde{\pi}} \to X$, we have

$$d_{4k-1}(nz_{(2k-1)j}) = \lambda y_{4k-1} \otimes z_{(2k-1)(j-2)}, \quad j = 2, 3, \cdots . \qquad (5)$$

Proof. G is the homotopy fiber of $f(\tilde{\pi})$. Diagram of fibrations:

$$\begin{array}{ccc} \Omega S^{2k} & \xrightarrow{\Omega[n]} & \Omega S^{2k} \\ \downarrow & & \downarrow \\ F_{\tilde{\pi}} & \longrightarrow & G \\ \downarrow & & \downarrow \\ X & \xrightarrow{p_X} & S^{4k-1} \end{array}$$

$$d_{4k-1}(z_{4k-2}) = H(f(\tilde{\pi}))\bar{y}_{4k-1} \quad .$$

$$(\Omega[n])^*(z_{4k-2}) = n^2 z_{4k-2}.$$

By the naturality of the Serre S.S.,

$d_{4k-1}((\Omega[n])^*(z_{4k-2})) = p_X^* d_{4k-1}(z_{4k-2})$, i.e.
$nd_{4k-1}(nz_{4k-2}) = H(f(\tilde{\pi}))y_{4k-1}$.
Thus we get $H(f(\tilde{\pi})) = \lambda n$ and $d_{4k-1}(nz_{4k-2}) = \lambda y_{4k-1}$.
The derivation-property of differential implies (5)

pf: $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n}$ $\Rightarrow S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr.

$$\begin{array}{ccccc} P^{2k}(n) & \xrightarrow{i_X} & X & \xrightarrow{p_X} & S^{4k-1} \\ \| & & \downarrow \tilde{\pi} & & \downarrow f(\tilde{\pi}) \\ P^{2k}(n) & \xrightarrow{p_{2k}} & S^{2k} & \xrightarrow{n\iota_{2k}} & S^{2k} \end{array} .$$

Prop 14. The Hopf invariant, $H(f(\tilde{\pi}))$, is a multiple of n, i.e. $H(f(\tilde{\pi})) = \lambda n$, and in the Serre S.S. for $\Omega S^{2k} \to F_{\tilde{\pi}} \to X$, we have

$$d_{4k-1}(nz_{(2k-1)j}) = \lambda y_{4k-1} \otimes z_{(2k-1)(j-2)}, \quad j = 2, 3, \cdots .$$

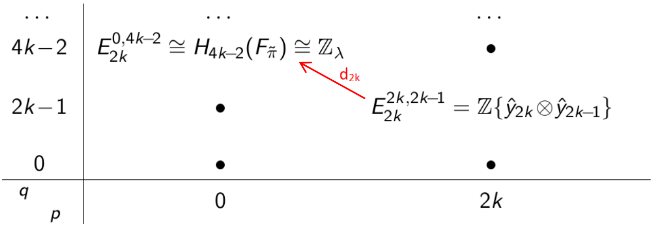
Lemma 17

If there is an extension $\tilde{\pi} : X \to S^{2k}$ of p_{2k}, then the reduced homology of $F_{\tilde{\pi}}$ are given by

$$\bar{H}_i(F_{\tilde{\pi}}) = \begin{cases} \mathbb{Z}\{\hat{y}_{2k-1}\}, & i = (2k-1); \\ \mathbb{Z}_\lambda\{\hat{y}_{(2k-1)j}\}, & i = (2k-1)j, j = 2, 3, \cdots; \\ 0, & \textit{otherwise.} \end{cases}$$

pf: $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n}$ $\Rightarrow S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr.

By the Serre S.S for the fib. $F_{\tilde{\pi}} \to X \to S^{2k}$ converging to $H_*(X)$:
$E^2_{p,q} = E^{2k}_{p,q} = H_p(S^{2k}) \otimes H_q(F_{\tilde{\pi}})$,



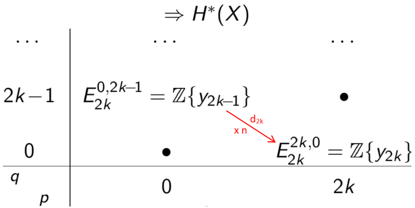
where $\hat{y}_{2k} \in H_{2k}(S^{2k})$ is a suitably chosen generator and $\hat{y}_{2k-1}$ is as in Lemma 17. Since $H_{4k-2}(X) = 0$, the differential $d_{2k} : E^{2k}_{2k,2k-1} \to E^{2k}_{0,4k-2} \cong H_{4k-2}(F_{\tilde{\pi}}) \cong \mathbb{Z}_\lambda$ must be an epimorphism. Hence the class $\lambda\hat{y}_{2k} \otimes \hat{y}_{2k-1} \in E^{\infty}_{2k,2k-1}$ represents a generator $[X] \in H_{4k-1}(X)$.

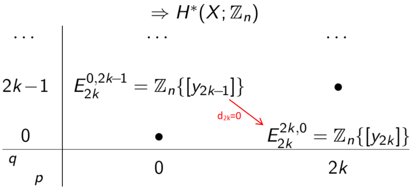
pf: $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n}$ $\Rightarrow S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr.

For the cohomology Serre S.S $\Rightarrow H^*(X), H^*(X, \mathbb{Z}_n)$

$E_2^{p,q} = E_{2k}^{p,q} = H^p(S^{2k}) \otimes H^q(F_{\tilde{\pi}})$, $\quad d_{2k}(y_{2k-1}) = ny_{2k}$

$E_2^{p,q} = E_{2k}^{p,q} = H^p(S^{2k}) \otimes H^q(F_{\tilde{\pi}}, \mathbb{Z}_n)$, $\quad d_{2k}([y_{2k-1}]) = n[y_{2k}] = 0$,





where $y_{2k} \in H^{2k}(S^{2k})$ is the class dual to $\hat{y}_{2k}$. The classes $[y_{2k-1}]$ and $[y_{2k}]$ are permanent cycles in this spectral sequence converging to $H^*(X, \mathbb{Z}_n)$. Note that $[y_{2k}]$ is the modulo-n reduction of an integral class in $H^{2k}(X)$, denoted by the same symbol for simplicity, and $\beta([y_{2k-1}]) = [y_{2k}]$.

pf: $i_*(f) = m[X_{2k}, \iota_{2k-1}]_r$, $m \equiv \pm\tau^2 \pmod{n} \Rightarrow S^{2k-1} \to X \xrightarrow{\pi} S^{2k}$ is a fibr.

By $\beta(\bar{e}^X_{2k-1}) = \bar{e}^X_{2k}$, if $\bar{e}^X_{2k-1} = \tau'[y_{2k-1}]$ with $\tau' \in \mathbb{Z}_n^*$, then $\bar{e}^X_{2k} = \tau'[y_{2k}]$.
From the definition of $[X]$ and Theorem 11, we have

$$\pm m = \left\langle \bar{e}^X_{2k-1} \cup \bar{e}^X_{2k}, [X] \right\rangle = \tau'^2 \langle [y_{2k-1}] \cup [y_{2k}], [X] \rangle \equiv \tau'^2 \lambda \pmod{n}$$

It follows from $m \equiv \pm\tau^2 \pmod{n}$, $\tau \in \mathbb{Z}_n^*$ that $\lambda \equiv \pm\tau_0^2 \pmod{n}$ with $\tau_0 = \tau_1\tau \in \mathbb{Z}_n^*$, where τ_1 is an integer such that $\tau_1 \equiv (\tau')^{-1} (\bmod\ n)$.

pf: $i_*(f)=m[X_{2k},\iota_{2k-1}]_r$, $m\equiv\pm\tau^2 \pmod{n}$ $\Rightarrow S^{2k-1}\to X\xrightarrow{\pi} S^{2k}$ is a fibr.

Let $n_2=\begin{cases}2n, & \text{if } k\neq 2,4;\\ n, & \text{if } k=2,4.\end{cases}$. If $2|n$ for $k\neq 2,4$, then $\tau_0\in\mathbb{Z}_n^*\subset\mathbb{Z}_{n_2}^*$. If $2\nmid n$ for $k\neq 2,4$, then $\lambda\equiv\pm(n-\tau_0)^2 \pmod{n}$, where $n-\tau_0\in\mathbb{Z}_n^*$. Replace τ_0 by $n-\tau_0$ if τ_0 is even. Then we always have $\lambda\equiv\pm\tau_0^2 \pmod{n}$ with τ_0 odd, i.e., $\tau_0\in\mathbb{Z}_{n_2}^*$. Take $\tau_2\equiv\tau_0^{-1}(\bmod\ n_2)$. Define $\pi: X\to S^{2k}$ as the composite $\pi=[\tau_2]\circ\bar{\pi}$. We get a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{q} & S^{4k-1} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle f(\pi)} \\ S^{2k} & \xrightarrow{[n]} & S^{2k} \end{array}$$

$$f(\pi)=[\tau_2]\circ f(\bar{\pi})$$

$$H(f(\pi))=\tau_2^2 H(f(\tilde{\pi}))=\pm n+n_2nt,\ t\in\mathbb{Z}.$$

Using the action of the group $\pi_{4k-1}(S^{2k})$ and Lemma 15, we may then assume $H(f(\pi))=\pm n$. It follows from Lemma 17 that the homotopy fiber of π is equivalent to S^{2k-1}, since it is a simply connected homology $(2k-1)$-sphere. This implies that X is the total space of an S^{2k-1} fibration over S^{2k} as required.

Thank You !