Simplicial Complexes with Extremal Total Betti Number and Total Bigraded Betti Number

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Content

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Section 1

Introduction

Total Betti number

For a finite CW-complex X and a field $\mathbb{F}$, let

$$\beta_i(X;\mathbb{F}) := \dim_{\mathbb{F}} H_i(X;\mathbb{F}), \quad \widetilde{\beta}_i(X;\mathbb{F}) := \dim_{\mathbb{F}} \widetilde{H}_i(X;\mathbb{F}).$$

$tb(X;\mathbb{F}) := \sum_i \beta_i(X;\mathbb{F})$ — total Betti number of X.

$\widetilde{tb}(X;\mathbb{F}) := \sum_i \widetilde{\beta}_i(X;\mathbb{F})$ — reduced total Betti number of X.

$\chi(X) := \sum_i (-1)^i \beta_i(X;\mathbb{F})$ — *Euler characteristic* of X.

$\widetilde{\chi}(X) := \sum_i (-1)^i \widetilde{\beta}_i(X;\mathbb{F})$ — *reduced Euler characteristic* of X.

The coefficient $\mathbb{F}$ will be omitted if there is no ambiguity in the context.

Total Betti number

The total Betti number of a topological space plays an important role in many theories in geometry and topology.

- Weak Morse Inequality: for a closed smooth manifold M and a Morse function $f : M \to \mathbb{R}$,

$$\# \text{ critical points of } f \geq tb(M; \mathbb{Z}_2).$$

- Smith Inequality: for a prime integer p and a $\mathbb{Z}_p$-action on a finite CW-complex X, the fixed point set $X^{\mathbb{Z}_p}$ satisfies:

$$tb(X^{\mathbb{Z}_p}; \mathbb{Z}_p) \leq tb(X; \mathbb{Z}_p).$$

- Halperin-Carlsson Conjecture: if a torus $T^k = (S^1)^k$ or a p-torus $(\mathbb{Z}_p)^k$ can act (almost) freely on a finite-dimensional CW-complex, then $tb(X; \mathbb{Q}) \geq 2^k$ or $tb(X; \mathbb{Z}_p) \geq 2^k$, respectively.

Simplicial Complex

Let K be a finite simplicial complex whose vertex set is

$$\mathrm{Ver}(K) = [m] = \{1, 2, \cdots, m\}.$$

Each simplex σ of K is considered as a subset of $[m]$.

Suppose $\dim(K) = d$.

The f-vector of K is $(f_0(K), f_1(K), \cdots, f_d(K))$ where $f_i(K)$ is the number of i-simplices in K.

The β-vector of K over a field $\mathbb{F}$ is

$$(\widetilde{\beta}_0(K;\mathbb{F}), \widetilde{\beta}_1(K;\mathbb{F}), \cdots, \widetilde{\beta}_d(K;\mathbb{F})).$$

Full subcomplex

For any subset $J \subseteq [m]$, let

$K|_J =$ the full subcomplex of K obtained by restricting to J.

In particular, when $J = \varnothing$, $K|_J = \varnothing$ and define

$$\beta_i(\varnothing) = 0,\ \forall i \geqslant 0;\ \ \widetilde{\beta}_i(\varnothing) = \begin{cases} 1, & \text{if } i = -1\ ; \\ 0, & \text{otherwise.} \end{cases}$$

Question 1

Question 1: For a positive integer m, which simplicial complexes have the maximum (reduced) total Betti number among all the simplicial complexes with m vertices?

When $m = 1, 2, 3$, the answer is just the discrete m points.

When $m = 4$, the answer is either the discrete 4 points or the complete graph on 4 vertices.

A complete answer to Question 1 has been obtained by Björner and Kalai in 1988.

A theorem of Björner and Kalai

Theorem [Björner-Kalai 1988]

Let K be a simplicial complex with at most $n+1$ vertices. Then

$$|\widetilde{\chi}(K)| \leqslant \widetilde{tb}(K) \leqslant \binom{n}{[n/2]}.$$

Moreover, the following conditions are equivalent:

(i) $|\widetilde{\chi}(K)| = \binom{n}{[n/2]}$,

(ii) $\widetilde{tb}(K) = \binom{n}{[n/2]}$,

(iii) K is the k-skeleton of an n-simplex, where $k = n/2 - 1$ if n is even and $k = (n-1)/2$ or $k = (n-3)/2$ if n is odd.

Remark: The proof of this Theorem uses a nontrivial operation called algebraic shifting of a simplicial complex and Sperner's theorem.

Question 2

Question 2: For each $0 \leqslant d < m$, which d-dimensional simplicial complexes with m vertices have the maximum total Betti number among all the d-dimensional simplicial complexes with m vertices?

For a pair of integers (m, d), $0 \leqslant d < m$, let

$\Sigma(m) =$ the set of all simplicial complexes with vertex set $[m]$.

$\Sigma(m, d) =$ the set of all d-dimensional simplicial complexes with vertex set $[m]$.

Question 1 and Question 2 are equivalent to determine the sets

$$\Sigma^{tb}(m) = \left\{ K \in \Sigma(m) \mid \widetilde{tb}(K) = \max_{L \in \Sigma(m)} \widetilde{tb}(L) \right\} \subseteq \Sigma(m).$$

$$\Sigma^{tb}(m, d) = \left\{ K \in \Sigma(m, d) \mid \widetilde{tb}(K) = \max_{L \in \Sigma(m,d)} \widetilde{tb}(L) \right\} \subseteq \Sigma(m, d).$$

Bigraded Betti numbers

For a simplicial complex $K \in \Sigma(m)$, the Stanley-Reisner ring of K over a commutative ring with unit R is

$$R[K] = R[v_1, \cdots, v_m]/\mathcal{I}_K$$

where $\mathcal{I}_K$ is the ideal generated by all the square-free monomials $v_{i_1} \cdots v_{i_s}$ where $\{i_1, \cdots, i_s\}$ is not a simplex of K.

Bigraded Betti numbers

By the standard construction in homological algebra, we obtain a canonical algebra $\mathrm{Tor}_{R[v_1,\cdots,v_m]}(R[K],R)$, where R is considered as the trivial $R[v_1,\cdots,v_m]$-module.

Moreover, there is a bigraded module structure on $\mathrm{Tor}_{R[v_1,\cdots,v_m]}(R[K],R)$

$$\mathrm{Tor}_{R[v_1,\cdots,v_m]}(R[K],R)=\bigoplus_{i,j\geq 0}\mathrm{Tor}^{-i,2j}_{R[v_1,\cdots,v_m]}(R[K],R)$$

where $\deg(v_i)=2$ for each $1\leqslant i\leqslant m$.

If R is a field $\mathbb{F}$, define

$$\beta^{-i,2j}(\mathbb{F}(K)):=\dim_{\mathbb{F}}\mathrm{Tor}^{-i,2j}_{\mathbb{F}[v_1,\cdots,v_m]}(\mathbb{F}[K],\mathbb{F})$$

called the bigraded Betti numbers of K with $\mathbb{F}$-coefficients.

Total bigraded Betti numbers

The total bigraded Betti number of K with $\mathbb{F}$-coefficients is

$$\widetilde{D}(K;\mathbb{F}) = \sum_{i,j} \beta^{-i,2j}(\mathbb{F}(K)) = \dim_{\mathbb{F}} \operatorname{Tor}_{\mathbb{F}[v_1,\cdots,v_m]}(\mathbb{F}[K],\mathbb{F}).$$

The Hochster's formula tells us that

$$\beta^{-i,2j}(\mathbb{F}(K)) = \sum_{J\subseteq[m],|J|=j} \dim_{\mathbb{F}} \widetilde{H}_{j-i-1}(K|_J;\mathbb{F}).$$

So we can also express $\widetilde{D}(K;\mathbb{F})$ as

$$\widetilde{D}(K;\mathbb{F}) = \sum_{J\subseteq[m]} \widetilde{tb}(K|_J;\mathbb{F}).$$

Question 3

Question 3: For each $0 \leqslant d < m$, which simplicial complexes in $\Sigma(m,d)$ have the minimum total bigraded Betti number over a field $\mathbb{F}$ among all the members in $\Sigma(m,d)$?

Such kind of simplicial complexes are called $\widetilde{D}$-minimal over $\mathbb{F}$.

Theorem [Cao-Lü 2011, Ustinovsky 2011]

For any $K \in \Sigma(m,d)$ and any field $\mathbb{F}$, $\widetilde{D}(K;\mathbb{F}) \geqslant 2^{m-d-1}$.

A simplicial complex $K \in \Sigma(m,d)$ is called tight over a field $\mathbb{F}$ if

$$\widetilde{D}(K;\mathbb{F}) = 2^{m-d-1}.$$

Remark: A $\widetilde{D}$-minimal simplicial complex is not necessarily tight.

Question 4 and Question 5

Question 4: For a positive integer m, which simplicial complexes in $\Sigma(m)$ have the maximum total bigraded Betti numbers among all the members in $\Sigma(m)$?

Equivalently: For what $K \in \Sigma(m)$ does $\dim_{\mathbb{F}} \mathrm{Tor}_{\mathbb{F}[v_1,\cdots,v_m]}(\mathbb{F}[K],\mathbb{F})$ reach the maximum?

Question 5: For each $0 \leqslant d < m$, which simplicial complexes in $\Sigma(m,d)$ have the maximum total bigraded Betti numbers among all the members in $\Sigma(m,d)$?

We can give a complete answer to Question 4. But we do not know the answer to Question 5.

Section 2

Main results

Some notations

For any $m \geq 1$, we use $\Delta^{[m]}$ to denote the $(m-1)$-dimensional simplex with vertex set $[m]$.

So $\partial\Delta^{[m]}$ is a simplicial sphere of dimension $m-2$.

Moreover, for any $0 \leqslant k < d < m$,

- let $\Delta^{[m]}_{(k)}$ denote the k-skeleton of $\Delta^{[m]}$;
- let $\Delta^{[m]}_{(k)}\langle d\rangle$ denote the minimal d-dimensional subcomplex of $\Delta^{[m]}$ that contains $\Delta^{[m]}_{(k)}$, which is unique up to simplicial isomorphism. Indeed, $\Delta^{[m]}_{(k)}\langle d\rangle$ is the union of $\Delta^{[m]}_{(k)}$ with a d-simplex.

Simplicial complexes with the maximal total Betti number in each dimension

We answer Question 2 in the following theorem.

Theorem 1 [Dai-Yu 2024]

The sets $\Sigma^{tb}(m,d)$ are classified as follows:

(i) If $d \leqslant \left[\frac{m}{2}\right] - 1$ or $d = m-1$, then $\Sigma^{tb}(m,d) = \left\{\Delta^{[m]}_{(d)}\right\}$;

(ii) If $\left[\frac{m}{2}\right] \leqslant d \leqslant m-3$, then $\Sigma^{tb}(m,d) = \left\{\Delta^{[m]}_{(\left[\frac{m}{2}\right]-1)}\langle d\rangle\right\}$;

(iii) If $d = m-2$,

- when m is odd, $\Sigma^{tb}(m,d) = \left\{\Delta^{[m]}_{(\left[\frac{m}{2}\right]-1)}\langle d\rangle, \Delta^{[m]}_{(\left[\frac{m}{2}\right])}\langle d\rangle\right\}$;
- when m is even, $\Sigma^{tb}(m,d) = \left\{\Delta^{[m]}_{(\left[\frac{m}{2}\right]-1)}\langle d\rangle\right\}$.

Classification of tight simplicial complexes

We classify all the tight simplicial complexes in the following theorem.

Theorem 2 [Dai-Yu 2024]

A finite simplicial complex K is tight if and only if K is of the form $\partial\Delta^{[n_1]} * \cdots * \partial\Delta^{[n_k]}$ or $\Delta^{[r]} * \partial\Delta^{[n_1]} * \cdots * \partial\Delta^{[n_k]}$ for some positive integers $n_1, \cdots, n_k$ and r.

Note that by convention, $\partial\Delta^{[1]} = \varnothing$ and $K * \varnothing = K$.

Remark: If $K \in \Sigma(m, d)$ is tight, it is necessary that $\left[\frac{m-1}{2}\right] \leq d$. The equality $\left[\frac{m-1}{2}\right] = d$ is achieved by $\partial\Delta^{[2]} * \partial\Delta^{[2]} * \cdots * \partial\Delta^{[2]}$ when m is even and by $\Delta^{[1]} * \partial\Delta^{[2]} * \partial\Delta^{[2]} * \cdots * \partial\Delta^{[2]}$ when m is odd.

Classification of tight simplicial complexes

So if $\left[\frac{m-1}{2}\right] \leq d \leq m-1$, the $\widetilde{D}$-minimal simplicial complexes are exactly all the tight simplicial complexes.

But when $\left[\frac{m-1}{2}\right] > d$, a $\widetilde{D}$-minimal simplicial complex in $\Sigma(m,d)$ is never tight.

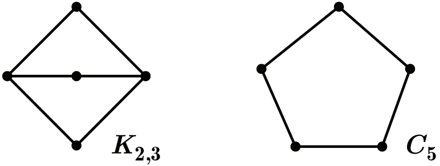


Figure 1: $\widetilde{D}$-minimal 1-dimensional simplicial complexes with 5 vertices

$\widetilde{D}$-minimal 1-dimensional simplicial complexes

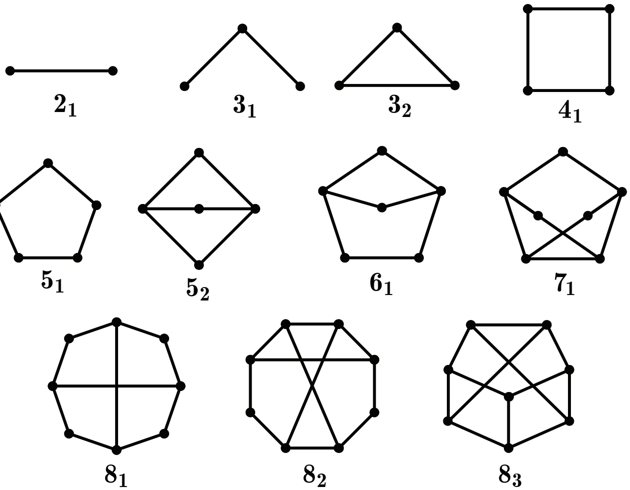


Figure 2: $\widetilde{D}$-minimal 1-dimensional simplicial complexes with ≤ 8 vertices

$\widetilde{D}$-minimal 1-dimensional simplicial complexes

- $\widetilde{D}(K_{2_1}) = 1.$ $\quad|\Sigma(2,1)| = 2$
- $\widetilde{D}(K_{3_1}) = \widetilde{D}(K_{3_2}) = 2.$ $\quad|\Sigma(3,1)| = 4$
- $\widetilde{D}(K_{4_1}) = 4.$ $\quad|\Sigma(4,1)| = 11$

- $\widetilde{D}(K_{5_1}) = \widetilde{D}(K_{5_2}) = 12 > 8.$ $\quad|\Sigma(5,1)| = 34$
- $\widetilde{D}(K_{6_1}) = 32 > 16.$ $\quad|\Sigma(6,1)| = 156$
- $\widetilde{D}(K_{7_1}) = 82 > 32.$ $\quad|\Sigma(7,1)| = 1044$
- $\widetilde{D}(K_{8_1}) = \widetilde{D}(K_{8_2}) = \widetilde{D}(K_{8_3}) = 196 > 64.$ $\quad|\Sigma(8,1)| = 12346$

It seems to us that there is no good way to describe all the $\widetilde{D}$-minimal simplicial complexes in $\Sigma(m,d)$ when $\left[\frac{m-1}{2}\right] > d$.

Simplicial complexes with the maximal total bigraded Betti number

We answer Question 4 in the following theorem. Let

$$g(m,d)=\sum_{j=d+1}^{m}\binom{m}{j}\binom{j-1}{d},\ 0\le d<m.$$

Theorem 3 [Dai-Yu 2024]

If K is a simplicial complex with m vertices, then for any field $\mathbb{F}$,

$$\widetilde{D}(K;\mathbb{F})\leqslant g\left(m,\left[\frac{m-1}{3}\right]\right)+1,$$

where the equality holds if and only if $K=\Delta^{[m]}_{([\frac{m-1}{3}]-1)}$.

Section 3

Outline of the proof

Simplicial complexes with the maximal total bigraded Betti number

Theorem 3 [Dai-Yu 2024]

If K is a simplicial complex with m vertices, then for any field $\mathbb{F}$,

$$\widetilde{D}(K;\mathbb{F}) \leqslant g\left(m, \left[\frac{m-1}{3}\right]\right) + 1,$$

where the equality holds if and only if $K = \Delta^{[m]}_{\left(\left[\frac{m-1}{3}\right]-1\right)}$.

$$g(m,d) = \sum_{j=d+1}^{m} \binom{m}{j}\binom{j-1}{d},\ 0 \leq d < m.$$

Proof of Theorem 3

We can easily show that if $\widetilde{D}(K;\mathbb{F})$ reaches the maximum, then K must be invariant under any permutation of its vertices, i.e. K is the d-skeleton of $\Delta^{[m]}$ for some d.

By an elementary calculation, we obtain

$$\widetilde{D}(\Delta^{[m]}_{(d)};\mathbb{F}) = \sum_{i=0}^{m-d-2} \binom{m}{m-i}\binom{m-i-1}{d+1} + 1 = g(m,d+1)+1.$$

Then Theorem 3 follows from the technical lemma below.

Lemma [Dai-Yu 2024]

For $0 \le d < m$, $g(m,d) = \sum_{j=d+1}^{m} \binom{m}{j}\binom{j-1}{d}$ reaches the maximum when and only when $d = \left[\frac{m-1}{3}\right]$.

Classification of Tight simplicial complexes

Theorem 2 [Dai-Yu 2024]

A finite simplicial complex K is tight if and only if K is of the form $\partial\Delta^{[n_1]} * \cdots * \partial\Delta^{[n_k]}$ or $\Delta^{[r]} * \partial\Delta^{[n_1]} * \cdots * \partial\Delta^{[n_k]}$.

For any positive integers $n_1, \cdots, n_k$ and r, call the simplicial complex $\partial\Delta^{[n_1]} * \cdots * \partial\Delta^{[n_k]}$ a sphere join and call $\Delta^{[r]} * \partial\Delta^{[n_1]} * \cdots * \partial\Delta^{[n_k]}$ a simplex-sphere join.

Theorem [Yu-Masuda 2022]

Let K be a simplicial complex of dimension $n \geq 2$. Suppose that K satisfies the following two conditions:

(a) K is an n-dimensional pseudomanifold,

(b) the link of any vertex of K is a sphere join of dimension $n-1$,

Then K is a sphere join.

Pseudomanifold

A simplicial complex K is called an n-dimensional pseudomanifold if the following conditions hold:

(i) Every simplex of K is a face of some n-simplex of K (i.e. K is pure).

(ii) Every $(n-1)$-simplex of K is the face of exactly two n-simplices of K.

(iii) If σ and σ' are two n-simplices of K, then there is a finite sequence of n-simplices $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ such that the intersection $\sigma_i \cap \sigma_{i+1}$ is an $(n-1)$-simplex for all $i = 0, \ldots, k-1$.

In particular, any closed connected PL-manifold is a pseudomanifold.

Proof of Theorem 2

Lemma [Dai-Yu 2024]

Let K be a simplicial complex with m vertices. If K is tight, then

(i) K is pure.

(ii) For every simplex σ of K, $\mathrm{Link}_K\sigma$ is tight.

(iii) If K is not connected, K must be S^0.

Suppose K is a tight simplicial complex with m vertices. Then by the above lemma and the induction on m, the link $\mathrm{Link}_K v$ of every vertex v of K is either a sphere-join or a simplex-sphere join. This implies that K is a PL-manifold (with boundary).

Proof of Theorem 2

- Case 1: The link of every vertex of K is a sphere-join. Then K is a closed PL-manifold, hence a pseudomanifold. So K must be a sphere-join by the above lemma.

- Case 2: There exists a vertex v of K with the link $\mathrm{Link}_K v$ being a simplex-sphere join. We can prove that there exists another vertex $w \in K$ such that $K = w * (K \backslash w)$ and $K \backslash w$ is also tight.

  By induction, $K \backslash w$ is a sphere-join or a simplex-sphere joint, then K is a simplex-sphere join. ■

Simplicial complexes with maximal total Betti number in each dimension

Theorem 1 [Dai-Yu 2024]

The sets $\Sigma^{tb}(m,d)$ are classified as follows:

(i) If $d \leqslant \left[\frac{m}{2}\right]-1$ or $d=m-1$, then $\Sigma^{tb}(m,d)=\left\{\Delta^{[m]}_{(d)}\right\}$;

(ii) If $\left[\frac{m}{2}\right] \leqslant d \leqslant m-3$, then $\Sigma^{tb}(m,d)=\left\{\Delta^{[m]}_{([\frac{m}{2}]-1)}\langle d\rangle\right\}$;

(iii) If $d=m-2$,

- when m is odd, $\Sigma^{tb}(m,d)=\left\{\Delta^{[m]}_{([\frac{m}{2}]-1)}\langle d\rangle, \Delta^{[m]}_{([\frac{m}{2}])}\langle d\rangle\right\}$;
- when m is even, $\Sigma^{tb}(m,d)=\left\{\Delta^{[m]}_{([\frac{m}{2}]-1)}\langle d\rangle\right\}$.

Shifted simplicial complex

Definition — Shifted Simplicial Complex

A simplicial complex Γ with vertex set $[m]$ is called *shifted* if for every simplex $\sigma = \{i_1, \cdots, i_s\} \in \Gamma$ where $i_1 < \cdots < i_s$, any $\{j_1, \cdots, j_s\}$ with $j_1 \leq i_1, \cdots, j_s \leq i_s$ and $j_1 < \cdots < j_s$ is also a simplex of Γ.

A shifting operation is a map which assigns to every simplicial complex K a shifted simplicial complex $\Delta(K)$ with the same f-vector.

A well-known shifting operation, was introduced by Erdös, Ko and Rado in 1961, also called combinatorial shifting, which has been of great use in extremal set theory.

Algebraic Shifting of a simplicial complex

Later in 1984, another shifting operation was introduced by Kalai called algebraic shifting, which preserves both the f-vector and the β-vector of a simplicial complex.

Theorem [Kalai 1984, Björner-Kalai 1988]

Given a simplicial complex K on m vertices and a field $\mathbb{F}$, there exists a canonically defined shifted simplicial complex $\Delta = \Delta(K, \mathbb{F})$ on $[m]$ such that

$$f_i(\Delta) = f_i(K), \quad \widetilde{\beta}_i(\Delta; \mathbb{F}) = \widetilde{\beta}_i(K; \mathbb{F}),\ i \geqslant 0.$$

Ordering of simplices

Let $\{i_1, i_2, \ldots, i_k\}_<$ denotes an ordered set where $i_1 < i_2 < \ldots < i_k$.

For two ordered sets $S = \{i_1, i_2, \ldots, i_k\}_<$ and $T = \{j_1, j_2, \ldots, j_k\}_<$ of the same size,

- the partial order is defined by:

$$S \leqslant_{\mathcal{P}} T \iff i_l \leqslant j_l \text{ for all } 1 \leqslant l \leqslant k;$$

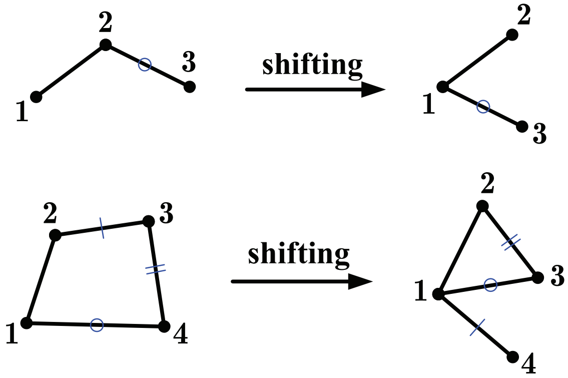
- the lexicographic order is defined by:

$$S \leqslant_{\mathcal{L}} T \iff S = T \text{ or } \min(S \,\Delta T) \in S,$$

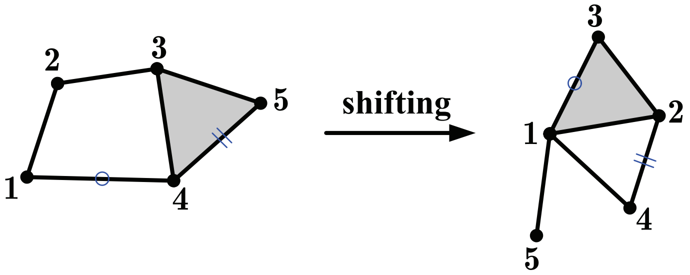
where $S \,\Delta T = (S \setminus T) \cup (T \setminus S)$ is the symmetric difference.

Example of shifting of a simplicial complex

Roughly speaking, algebraic shifting a simplicial complex is:
starting from lower dimension to higher dimension, nudging all the simplices forward with respect to the lexicographic order.



Example of shifting of a simplicial complex



Remark: Algebraic shifting may not preserve the homotopy type of a simplicial complex and may not even induce a chain map.

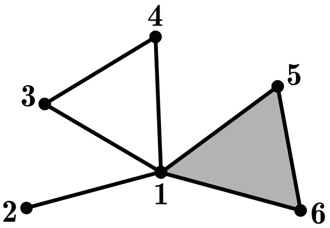
Near-cone

In fact, the shifted complex $\Delta(K)$ associated to K belongs to a slightly larger class of simplicial complexes called near-cones.

Definition — Near-Cone

A simplicial complex Δ with vertex set $[m]$ is called a near-cone if for any simplex $S \in \Delta$ and $j \geq 2$,

$$\text{if } 1 \notin S \text{ and } j \in S\text{, then } (S \setminus j) \cup \{1\} \in \Delta.$$



Property of Near-cone

For a near-cone Δ, define

$$B(\Delta) = \{S \in \Delta \mid S \cup \{1\} \notin \Delta\}.$$

A very nice property of a near-cone Δ is: $\widetilde{tb}(\Delta) = |B(\Delta)|$.

Lemma [Björner-Kalai 1988]

If Δ is a near-cone on $[m]$, then

(i) every simplex $S \in B(\Delta)$ is maximal in Δ,

(ii) Δ is homotopy equivalent to a wedge of spheres

$$\Delta \simeq \bigvee_{0 \le i \le \dim(\Delta)} \bigvee_{f_i(B(\Delta))} S^i \implies \widetilde{tb}(\Delta) = |B(\Delta)|,$$

(iii) $B(\Delta)$ is a Sperner family of $\{2, \cdots, m\}$.

Sperner's Theorem

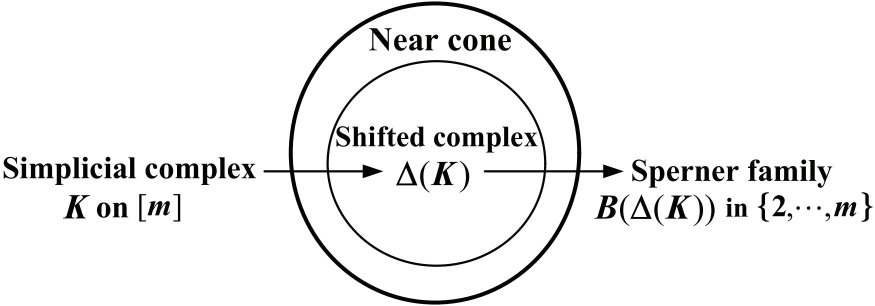
Definition — Sperner Family

Let X be a finite set. A *Sperner family* of X is a set $\mathcal{F}$ of subsets of X that satisfies $A \nsubseteq B$ for distinct members of $\mathcal{F}$. Given a subset $Y \subseteq X$, a Sperner family of X over Y is a Sperner family $\mathcal{F}$ of X where every member of $\mathcal{F}$ has nonempty intersection with Y.

Theorem [Sperner 1928]

Let $\mathcal{F}$ be a Sperner family of subsets of a finite set X where $|X| = n$. Then $|\mathcal{F}| \leq \binom{n}{[n/2]}$. If n is even, the only Sperner family consisting of $\binom{n}{[n/2]}$ subsets of X is made up of all the $\frac{n}{2}$-subsets of X. If n is odd, a Sperner family of size $\binom{n}{[n/2]}$ consists of either all the $\frac{1}{2}(n-1)$-subsets or all the $\frac{1}{2}(n+1)$-subsets of X.

Summary



$$\widetilde{tb}(K)=\widetilde{tb}(\Delta(K))=|B(\Delta(K))|$$

Property of Near-cone

The following proposition tells us what kind of Sperner families on $\{2, \cdots, m\}$ are in the form of $B(\Delta)$ for some d-dimensional near-cone Δ on $[m]$.

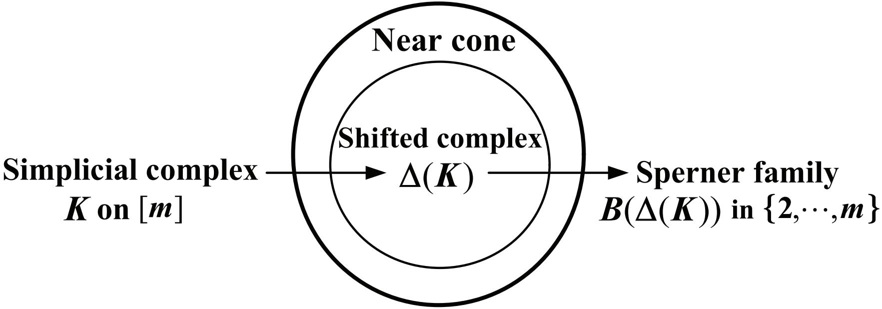
Proposition [Dai-Yu 2024]

Let $\mathcal{F}$ be a Sperner family of $[m]\backslash\{1\} = \{2, \cdots, m\}$. The following statements are equivalent:

(i) there exists a d-dimensional near-cone Δ with vertex set contained in $[m]$, such that $B(\Delta) = \mathcal{F}$;

(ii) there exists a subset $\{i_1, \cdots, i_d\} \subseteq \{2, \cdots, m\}$ such that $\mathcal{F}$ is a Sperner family of $\{2, \cdots, m\}$ over $\{2, \cdots, m\}\backslash\{i_1, \cdots, i_d\}$ and the order of each member of $\mathcal{F}$ is no greater than $d + 1$.

Summary



$$\widetilde{tb}(K)=\widetilde{tb}(\Delta(K))=|B(\Delta(K))|$$

$\dim(K)=d$, assume $\{1,\cdots,d+1\}$ is a simplex of K $\Rightarrow$ $B(\Delta(K))$ is a Sperner family in $\underbrace{\{2,\cdots,m\}}_{X}$ over $\underbrace{\{d+2,\cdots,m\}}_{Y}$

Sperner families with the maximal cardinality

Let X be a set with order $|X| = n$. For a nonempty subset Y of X, let

$C(n, X, Y) =$ the set of all subsets of X that have nonempty intersection with Y.

For any $i \geq 1$, let

$C_i(n, X, Y) =$ the collection of sets in $C(n, X, Y)$ of size i.

Sperner families with the maximal cardinality

Theorem [Lih 1980 + Griggs 1982]

Let X be a finite set of order n. The maximal possible cardinality of a Sperner family $\mathcal{F}$ of X over a subset $Y \subseteq X$ with $|Y| = k$ is $f(n,k) = \binom{n}{\lceil n/2 \rceil} - \binom{n-k}{\lceil n/2 \rceil}$. Moreover, $|\mathcal{F}| = f(n,k)$ if and only if $\mathcal{F}$ is one of the following cases:

(a) $C_{\lceil \frac{1}{2}n \rceil}(n, X, Y)$ where $\lceil \cdot \rceil$ is the ceiling function;

(b) $C_{\frac{1}{2}(n-1)}(n, X, Y)$, for odd n and $k \geqslant \frac{1}{2}(n+3)$;

(c) $C_{\frac{1}{2}(n+2)}(n, X, Y)$, for even n and $k = 1$.

In particular, if $|\mathcal{F}| = f(n,k)$, every member in $\mathcal{F}$ has the same order.

Remark: Not all Sperner families with the maximal cardinality in the above theorem are of the form $B(\Delta)$ for some near-cone Δ.

Proof of Theorem 1

- By an induction on the dimension d, we can prove:

(i) If $d \leqslant \left[\frac{m}{2}\right] - 1$ or $d = m - 1$, then $\Sigma^{tb}(m,d) = \left\{\Delta^{[m]}_{(d)}\right\}$;

- By figuring out all the possible near-cones corresponding to the Sperner families with the maximal cardinality in the above theorem, we obtain

(ii) If $\left[\frac{m}{2}\right] \leqslant d \leqslant m - 3$, then $\Sigma^{tb}(m,d) = \left\{\Delta^{[m]}_{(\left[\frac{m}{2}\right]-1)}\langle d\rangle\right\}$;

(iii) If $d = m - 2$,

  - when m is odd, $\Sigma^{tb}(m,d) = \left\{\Delta^{[m]}_{(\left[\frac{m}{2}\right]-1)}\langle d\rangle, \Delta^{[m]}_{(\left[\frac{m}{2}\right])}\langle d\rangle\right\}$;
  - when m is even, $\Sigma^{tb}(m,d) = \left\{\Delta^{[m]}_{(\left[\frac{m}{2}\right]-1)}\langle d\rangle\right\}$.

Thank you for your attention

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