Self-covering, finiteness, fibering over tori

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Self-covering manifolds

A manifold M is (homotopy) self-covering if it is CAT-isomorphic (homotopy equivalent) to a non-trivial covering of itself.

$$M \xrightarrow{h} M' \xrightarrow{q} M$$

Low dimensional self-covering manifolds:

1. $\dim \leq 2$: S^1, $\mathbb{T}^2$ and the Klein bottle.
2. $\dim = 3$

Theorem (Wang-Yu)

Let M be a closed 3-manifold. Then M is homeomorphic to a nontrivial cover of itself if and only if it is finitely covered by a $\mathbb{T}^2$-bundle or a trivial surface bundle over S^1.

Main question: fibering over tori

Examples.

$A\colon \mathbb{T}^n \to \mathbb{T}^n$, $A \in \mathrm{End}(\mathbb{Z}^n)$, $\det A \neq 0$, $\bigcap_{k=1}^{\infty} \mathrm{Im} A^k = 0$.

More general examples: $(\mathbb{T}^n \times N, A \times \mathrm{id}_N)$.

The structure of self-covering manifolds with abelian $\pi_1(M)$: *fiber bundle over* $\mathbb{T}^n$ or *fiber bundle with fiber* $\mathbb{T}^n$*?*

Question

Give a self-covering manifold $M \xrightarrow{h} M' \xrightarrow{q} M$ *with abelian* $\pi_1(M)$*, is there a CAT-bundle projection* $p\colon M \to \mathbb{T}^n$ *and* $A \in \mathrm{End}(\mathbb{Z}^n)$ *with* $\det A \neq 0$, $\bigcap_{k=1}^{\infty} \mathrm{Im} A^k = 0$ *such that*

$$\begin{array}{ccccc} M & \xrightarrow{h} & M' & \xrightarrow{q} & M \\ & \searrow^{p} & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ & & \mathbb{T}^n & \xrightarrow{A} & \mathbb{T}^n \end{array}$$

Towards an answer to the Question: Step 1 — the base $\mathbb{T}^n$

Given a self-covering $M \xrightarrow{h} M' \xrightarrow{q} M$, how to find $\mathbb{T}^n$?

Consider $h_\# : \pi_1(M) \hookrightarrow \pi_1(M)$, define the residue group

$$G := \bigcap_{k=1}^{\infty} \mathrm{Im}\, h_\#^k$$

Theorem

$\pi_1(M)/G$ is isomorphic to $\mathbb{Z}^n$ for some $n \geq 1$.

Consequences:

- The quotient map $\pi_1(M) \to \mathbb{Z}^n$ induces a map $p\colon M \to \mathbb{T}^n$.
- The homomorphism $h_\# : \pi_1(M) \hookrightarrow \pi_1(M)$ induces a homomorphism $A\colon \mathbb{Z}^n \to \mathbb{Z}^n$ s.t. $\det A \neq 0$, $\bigcap_{k=1}^{\infty} \mathrm{Im}\, A^k = 0$.

Towards an answer to the Question: Step 2 – homotopy fiber

From Step 1 we have a map $p\colon M \to \mathbb{T}^n$ with $\operatorname{Ker} p_\# = G$

Let $\overline{M} \to M$ be the covering space with $\pi_1(\overline{M}) = G$. Then $\overline{M} \simeq \mathrm{hfib}(p)$.

If p is homotopic to a bundle projection, then $\overline{M}$ is homotopy equivalent to a finite CW-complex satisfying Poincaré duality.

Theorem (Key Theorem)

$\overline{M}$ is homotopy equivalent to a finitely dominated complex, satisfying Poincaré duality of dimension $m - n$.

A space X is finitely dominated if there exist a finite CW-complex Y and maps $X \xrightarrow{i} Y \xrightarrow{r} X$ such that $r \circ i \simeq \mathrm{id}_X$.

Towards an answer to the Question: Step 2 – homotopy fiber

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Theorem (Gottlieb, 1979)

Let $F \to E \to B$ be a fibration with both B and F being finitely dominated. Then E is a Poincaré duality space if and only both B and F are.

Application: a characterization of tori

Corollary

Let M be an m-dimensional closed TOP-manifold with abelian $\pi_1(M)$. If there is a self-covering $M \xrightarrow{h} M' \xrightarrow{q} M$ with residue group $G = \bigcap_{k=1}^{\infty} \operatorname{Im} h_{\#}^{k}$, $n = \operatorname{rank}(\pi_1(M)/G)$. If one of the followings holds

1. $m - n \leq 1$;
2. $m - n = 2$ *and* $|G| > 2$;
3. $m - n = 3$, *G is not cyclic and* $G \neq \mathbb{Z} \oplus \mathbb{Z}/2$.

Then M is homeomorphic to $\mathbb{T}^m$.

Remark

Farrell-Jones (1978): There are exotic tori with smooth expanding maps.

Application: a characterization of tori

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Proof.

We have a fibration $\overline{M} \to M \to \mathbb{T}^n$ with $\dim \overline{M} = m - n$.
If $m - n \leq 1$, then $\overline{M} \simeq *$ or S^1. Hence $M \simeq \mathbb{T}^m$. Topological rigidity $\Rightarrow M \cong T^m$. $\square$

Application: 4-dimensional self-covering manifolds

Theorem

Let M be a closed TOP 4-manifold with abelian $\pi_1(M)$, and M is homotopy equivalent to a nontrivial cover of itself. Then M satisfies one of the followings:

1. $\pi_1(M) = \mathbb{Z}$, $M \cong S^1 \times S^3$ or $S^1 \tilde{\times} S^3$;
2. $\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}/2$, $M \simeq S^1 \times \mathbb{R}P^3$ or $S^1 \tilde{\times} \mathbb{R}P^3$;
3. $\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}/k$ $(k > 2)$, $M \simeq S^1 \times L$;
4. $\pi_1(M) = \mathbb{Z}^2$, $M \cong \mathbb{T}^2 \times S^2$, $S^1 \times (S^1 \tilde{\times} S^2)$, E_1 or E_2;
5. $\pi_1(M) = \mathbb{Z}^2 \oplus \mathbb{Z}/2$, $M \cong \mathbb{T}^2 \times \mathbb{R}P^2$ or PE_1;
6. $\pi_1(M) = \mathbb{Z}^4$, $M \cong \mathbb{T}^4$.

High dimensions: a weaker answer — fibering over S^1

Theorem

Let M be a closed CAT-manifold of dimension $m \geq 6$ with free abelian $\pi_1(M)$. If there is a self-covering $M \xrightarrow{h} M' \xrightarrow{q} M$. Let $G = \bigcap_{k=1}^{\infty} \operatorname{Im} h_{\#}^{k}$. Then for any surjective homomorphism

$$\theta \colon \pi_1(M) \to \pi_1(M)/G = \mathbb{Z}^n \twoheadrightarrow \mathbb{Z}$$

there is a CAT-bundle projection $p \colon M \to S^1$ with $p_{\#} = \theta$.

Proof.

There exists a map $p \colon M \to S^1$ with $p_{\#} = \theta$. By the Key Theorem, the homotopy fiber of p is finitely dominated. Then the theorem follows from Farrell's Fibration Theorem. □

Remark

If $\pi_1(M)$ contains torsion, then there are counter-examples.

Towards an answer to the Question: Step 3 – block bundle

Let E and F be CAT-manifolds, K be a simplicial complex, $p\colon E \to |K|$ a continuous map. Suppose for any simplex $\sigma \in K$ with faces $\partial_0\sigma, \cdots, \partial_n\sigma$, there is a CAT-isomorphism

$$\Phi\colon (\sigma\times F; \partial_0\sigma\times F, \cdots, \partial_n\sigma\times F) \to (p^{-1}(\sigma); p^{-1}(\partial_0\sigma), \cdots, p^{-1}(\partial_n\sigma)).$$

Then we call $p\colon E \to |K|$ a CAT block bundle with fiber F.

Theorem (block bundle theorem)

Let M be a closed TOP manifold of dimension m with free abelian $\pi_1(M)$. If there is a self-covering $M \xrightarrow{h} M' \xrightarrow{q} M$. Let $G = \bigcap_{k=1}^{\infty} \mathrm{Im}\, h_{\#}^{k}$ with $\pi_1(M)/G = \mathbb{Z}^n$. Then $p\colon M \to \mathbb{T}^n$ is homotopic to a TOP block bundle projection if $m - n \geq 5$.

From block bundle to fiber bundle

$A(F)$ = the CAT-automorphism group of F

$\widetilde{A}(F)$ = the block CAT-automorphism group of F.

$\widetilde{A}(F)$ is a semi-simplicial group, an n-simplex in $\widetilde{A}(F)$ is a face-preserving CAT-automorphism

$$\Delta^n \times F \to \Delta^n \times F$$

The lifting problem

$$\begin{array}{ccc} & & BA(F) \\ & \overset{?}{\nearrow} & \downarrow \\ \mathbb{T}^n & \longrightarrow & B\widetilde{A}(F) \end{array}$$

has obstructions $o_i \in H^i(\mathbb{T}^n; \pi_{i-1}\widetilde{A}(F)/A(F))$.

Special case 1: low dimensional base

Theorem

Under the assumption of block bundle theorem. If $G = 0$ and $n = 1$ or 2, $m \geq 7$. Then the map $p\colon M \to \mathbb{T}^n$ is homotopic to a TOP bundle projection.

Proof.

Hatcher's spectral sequence

$$E^1_{ij} = \pi_j(C(F \times [0,1]^i)) \Rightarrow \pi_{i+j+1}(\widetilde{A}(F)/A(F))$$

where $C(F) = A(F \times I, F \times 0)$ is the concordance group of F.

If $G = \pi_1(F) = 0$ and $\dim F \geq 4$, then $C(F)$ is path connected. Therefore $\pi_1(\widetilde{A}(F)/A(F)) = 0$, $H^2(\mathbb{T}^2; \pi_1(\widetilde{A}(F)/A(F))) = 0$. □

Special case 2: highly connected fiber

Theorem

Under the assumption of block bundle theorem. If $G = 0$, $\pi_i(M) = 0$ for $2 \leq i \leq r$, where $m - n \geq r + 4$ and $n \leq \min\{2r - 1, r + 4\}$. Then the map $p\colon M \to \mathbb{T}^n$ is homotopic to a TOP bundle projection with r-connected fiber.

Example. If $r = 2$, then $m - n \geq 6$, $n \leq 3$.

Proof.

- The fiber F is r-connected, $\dim F = f \geq r + 4$.
- For $j \leq n - 1$, $\pi_j(\widetilde{A}(F)/A(F), \widetilde{A}_\partial(D^f)/A_\partial(D^f)) = 0$
- Alexander's trick $\Rightarrow \widetilde{A}_\partial(D^f)/A_\partial(D^f) \simeq *$.
- Therefore $H^i(\mathbb{T}^n; \pi_{i-1}\widetilde{A}(F)/A(F)) = 0$ for all $i \leq n$.

$\square$

Special case 3: stabilization

Theorem

Under the assumption of block bundle theorem. Let $s = n(n-1)/2$*. Then*

$$M \times \mathbb{T}^s \xrightarrow{\mathrm{pr}_1} M \xrightarrow{p} \mathbb{T}^n$$

is homotopic to a TOP bundle projection.

Proof.

A direct consequence of Burghelea-Lashof-Rothenberg. □

Non-fibering examples

Theorem

For any $n \geq 4$, $i = 2, 3$, there exists a closed TOP manifold M such that

1. *$M \simeq \mathbb{T}^n \times S^i$ (hence M is homotopy self-covering);*
2. *M is not a TOP bundle over $\mathbb{T}^n$.*

If $n \geq 8$, M is a smooth manifold.

Proof.

If $p\colon M \to \mathbb{T}^n$ is a fiber bundle, then the fiber is S^i, with structure group $\mathrm{Homeo}(S^i) \simeq O(i+1)$, hence is the sphere bundle of a rank $i+1$ vector bundle. Then $TM = \zeta \oplus p^* T\mathbb{T}^n$, $p_2(M) = 0$.

Starting from a degree 1 normal map $f\colon N \to \mathbb{T}^n$ with non-trivial surgery obstruction. For $f \times \mathrm{id}\colon N \times S^i \to \mathbb{T}^n \times S^i$ the surgery obstruction is trivial, we get $M \simeq \mathbb{T}^n \times S^i$ with $p_2(M) \neq 0$ $\square$

Non-abelian fundamental group: fibering problem

A self-covering manifold $M \xrightarrow{h} M' \xrightarrow{q} M$ with non-abelian π_1

Question

Is there a bundle projection $p\colon M \to B$ such that

$$\begin{array}{ccccc} M & \xrightarrow{h} & M' & \xrightarrow{q} & M \\ & \searrow^{p} & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ & & B & \xrightarrow{\psi} & B \end{array}$$

where B is a closed aspherical manifold, $\psi\colon B \to B$ a covering map such that $\bigcap_{k=1}^{\infty} \mathrm{Im}\,\psi_{\#}^{k} = 0$.

Define the residue group $G = \bigcap_{k=1}^{\infty} \mathrm{Im}\, h_{\#}^{k}$, then

$$\pi_1(\mathrm{hfib}(p)) = G = \mathrm{Ker} p_{\#}$$

Non-abelian fundamental group: a non-fibering example

Necessary conditions:

1. G is a normal subgroup of $\pi_1(M)$;
2. G is finitely presented.

Theorem

Let $BS(2,3) = \langle a, t \mid ta^2t^{-1} = a^3 \rangle$. Then for each $m \geq 5$, there exists a smooth closed m-dimensional manifold M with $\pi_1(M) = BS(2,3)$, such that there is a 5-fold covering map $q \colon M \to M$, with $G = \cap_{k=1}^{\infty} \mathrm{Im} q_{\#}^{k}$. The followings hold

1. *G is not a normal subgroup of $\pi_1(M)$;*
2. *G is isomorphic to F_∞, hence not finitely generated.*

From CW-complexes to manifolds

- $X = K(BS(2,3),1)$ a finite 2-dimensional complex
- there is a 5-fold covering $X' \to X$ such that $\pi_1(X') = \langle a^5, t\rangle < BS(2,3)$
- there is a homeomorphism $h\colon X \to X'$ with $h_\#(a) = a^{-5}$, $h_\#(t) = t$
- $G = \bigcap_{k=1}^{\infty} \mathrm{Im} h_\#^k \cong F_\infty$ is not a normal subgroup of $BS(2,3)$

Theorem

Let X be a finite 2-dimensional CW complex, X' be a k-fold covering of X. If there is a homeomorphism $h\colon X \to X'$. Then there exists a compact smooth hypersurface M in $\mathbb{R}^6$ with $\pi_1(M) = \pi_1(X)$. Let M' be the k-fold covering of M with $\pi_1(M') = \pi_1(X')$, there is a diffeomorphism $\varphi\colon M \to M'$ such that $h_\# = \varphi_\#$.

Finiteness: homotopy

Given a map $p\colon M \to \mathbb{T}^n$, how to show that $\mathrm{hfib}(p) \simeq \overline{M}$ is finitely dominated?

Theorem (Wall)

Let X be a CW-complex with finitely presented $\pi_1(X)$ and $\Lambda = \mathbb{Z}[\pi_1(X)]$ a noetherian ring. Then X is finitely dominated if and only if $H_i(X;\Lambda)$ is a finite Λ-module for all i and the cohomological dimension of X is finite.

Finiteness: algebra

We need to show that $H_i(\overline{M}; \mathbb{Z}[G]) = H_i(\widetilde{M}; \mathbb{Z})$ is a finite $\mathbb{Z}[G]$-module.

By construction $H_i(\overline{M}; \mathbb{Z}[G]) = H_i(\widetilde{M}; \mathbb{Z})$ is a finite $\mathbb{Z}[\pi_1(M)]$-module, with $\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[G][\mathbb{Z}^n]$

Theorem

Let R be a commutative noetherian ring, $S = R[t_1^{\pm 1}, \cdots, t_n^{-1}]$. Assume there is a ring monomorphism $\phi\colon S \to S$ with $\phi(R) = R$, and $\phi(t_j) = g_j \prod_{i=1}^n t_i^{a_{ij}}$, where $g_j \in R^\times$ and $A = (a_{ij})_{n\times n}$ satisfies $\det A \neq 0$, $\bigcap_{k=1}^\infty \operatorname{Im} A^k = 0$. If $\mathfrak{M}$ is a finite S-module and there is a ϕ-twisted S-module isomorphism $\eta\colon \mathfrak{M} \to \mathfrak{M}$, i.e., $\eta(sx) = \phi(s)\eta(x)$, then $\mathfrak{M}$ is a finite R-module.

Thank You