Self-covering, finiteness, fibering over tori

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Self-covering manifolds

A manifold M is (homotopy) self-covering if it is CAT-isomorphic (homotopy equivalent) to a non-trivial covering of itself.

$$M \stackrel{h}{\longrightarrow} M' \stackrel{q}{\longrightarrow} M$$

Low dimensional self-covering manifolds:

- 1. dim \leq 2: S^1 , \mathbb{T}^2 and the Klein bottle.
- $2. \ dim = 3$

Theorem (Wang-Yu)

Let M be a closed 3-manifold. Then M is homeomorphic to a nontrivial cover of itself if and only if it is finitely covered by a \mathbb{T}^2 -bundle or a trivial surface bundle over S^1 .

Main question: fibering over tori

Examples.

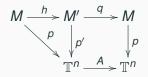
 $A: \mathbb{T}^n \to \mathbb{T}^n$, $A \in \text{End}(\mathbb{Z}^n)$, $\det A \neq 0$, $\bigcap_{k=1}^{\infty} \text{Im} A^k = 0$.

More general examples: $(\mathbb{T}^n \times N, A \times id_N)$.

The structure of self-covering manifolds with abelian $\pi_1(M)$: fiber bundle over \mathbb{T}^n or fiber bundle with fiber \mathbb{T}^n ?

Question

Give a self-covering manifold $M \stackrel{h}{\longrightarrow} M' \stackrel{q}{\longrightarrow} M$ with abelian $\pi_1(M)$, is there a CAT-bundle projection $p \colon M \to \mathbb{T}^n$ and $A \in \operatorname{End}(\mathbb{Z}^n)$ with $\det A \neq 0$, $\bigcap_{k=1}^{\infty} \operatorname{Im} A^k = 0$ such that



Towards an answer to the Question: Step 1 — the base \mathbb{T}^n

Given a self-covering $M \xrightarrow{h} M' \xrightarrow{q} M$, how to find \mathbb{T}^n ?

Consider $h_{\#} : \pi_1(M) \hookrightarrow \pi_1(M)$, define the residue group

$$G:=\bigcap_{k=1}^{\infty}\operatorname{Im}h_{\#}^{k}$$

Theorem

 $\pi_1(M)/G$ is isomorphic to \mathbb{Z}^n for some $n \geq 1$.

Consequences:

- The quotient map $\pi_1(M) \to \mathbb{Z}^n$ induces a map $p \colon M \to \mathbb{T}^n$.
- The homomorphism $h_{\#} \colon \pi_1(M) \hookrightarrow \pi_1(M)$ induces a homomorphism $A \colon \mathbb{Z}^n \to \mathbb{Z}^n$ s.t. $\det A \neq 0$, $\bigcap_{k=1}^{\infty} \operatorname{Im} A^k = 0$.

Towards an answer to the Question: Step 2 – homotopy fiber

From Step 1 we have a map $p: M \to \mathbb{T}^n$ with $\operatorname{Ker} p_\# = G$ Let $\overline{M} \to M$ be the covering space with $\pi_1(\overline{M}) = G$. Then $\overline{M} \simeq \operatorname{hfib}(p)$.

If p is homotopic to a bundle projection, then \overline{M} is homotopy equivalent to a finite CW-complex satisfying Poincaré duality.

Theorem (Key Theorem)

 \overline{M} is homotopy equivalent to a finitely dominated complex, satisfying Poincaré duality of dimension m-n.

A space X is finitely dominated if there exist a finite CW-complex Y and maps $X \stackrel{i}{\longrightarrow} Y \stackrel{r}{\longrightarrow} X$ such that $r \circ i \simeq \mathrm{id}_X$.

Towards an answer to the Question: Step 2 – homotopy fiber

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Theorem (Gottlieb, 1979)

Let $F \to E \to B$ be a fibration with both B and F being finitely dominated. Then E is a Poincaré duality space if and only both B and F are.

Application: a characterization of tori

Corollary

Let M be an m-dimensional closed TOP-manifold with abelian $\pi_1(M)$. If there is a self-covering $M \stackrel{h}{\longrightarrow} M' \stackrel{q}{\longrightarrow} M$ with residue group $G = \bigcap_{k=1}^{\infty} \operatorname{Im} h_{\#}^k$, $n = \operatorname{rank}(\pi_1(M)/G)$. If one of the followings holds

- 1. $m n \le 1$;
- 2. m n = 2 and |G| > 2;
- 3. m-n=3, G is not cyclic and $G \neq \mathbb{Z} \oplus \mathbb{Z}/2$.

Then M is homeomorphic to \mathbb{T}^m .

Remark

Farrell-Jones (1978): There are exotic tori with smooth expanding maps.

Application: a characterization of tori

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Proof.

We have a fibration $\overline{M} \to M \to \mathbb{T}^n$ with dim $\overline{M} = m - n$. If $m - n \le 1$, then $\overline{M} \simeq *$ or S^1 . Hence $M \simeq \mathbb{T}^m$. Topological rigidity $\Rightarrow M \cong T^m$.

Application: 4-dimensional self-covering manifolds

Theorem

Let M be a closed TOP 4-manifold with abelian $\pi_1(M)$, and M is homotopy equivalent to a nontrivial cover of itself. Then M satisfies one of the followings:

- 1. $\pi_1(M) = \mathbb{Z}$, $M \cong S^1 \times S^3$ or $S^1 \widetilde{\times} S^3$;
- 2. $\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}/2$, $M \simeq S^1 \times \mathbb{R}P^3$ or $S^1 \widetilde{\times} \mathbb{R}P^3$;
- 3. $\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}/k \ (k > 2), M \simeq S^1 \times L;$
- 4. $\pi_1(M) = \mathbb{Z}^2$, $M \cong \mathbb{T}^2 \times S^2$, $S^1 \times (S^1 \widetilde{\times} S^2)$, E_1 or E_2 ;
- 5. $\pi_1(M) = \mathbb{Z}^2 \oplus \mathbb{Z}/2$, $M \cong \mathbb{T}^2 \times \mathbb{R}P^2$ or PE_1 ;
- 6. $\pi_1(M) = \mathbb{Z}^4$, $M \cong \mathbb{T}^4$.

High dimensions: a weaker answer — fibering over S^1

Theorem

Let M be a closed CAT-manifold of dimension $m \ge 6$ with free abelian $\pi_1(M)$. If there is a self-covering $M \stackrel{h}{\longrightarrow} M' \stackrel{q}{\longrightarrow} M$. Let $G = \bigcap_{k=1}^{\infty} \operatorname{Im} h_{\#}^k$. Then for any surjective homomorphism

$$\theta \colon \pi_1(M) \to \pi_1(M)/G = \mathbb{Z}^n \twoheadrightarrow \mathbb{Z}$$

there is a CAT-bundle projection $p: M \to S^1$ with $p_\# = \theta$.

Proof.

There exists a map $p: M \to S^1$ with $p_\# = \theta$. By the Key Theorem, the homotopy fiber of p is finitely dominated. Then the theorem follows from Farrell's Fibration Theorem.

Remark

If $\pi_1(M)$ contains torsion, then there are counter-examples.

Towards an answer to the Question: Step 3 – block bundle

Let E and F be CAT-manifolds, K be a simplicial complex, $p \colon E \to |K|$ a continuous map. Suppose for any simplex $\sigma \in K$ with faces $\partial_0 \sigma, \cdots, \partial_n \sigma$, there is a CAT-isomorphism

$$\Phi \colon (\sigma \times F; \partial_0 \sigma \times F, \cdots, \partial_n \sigma \times F) \to (p^{-1}(\sigma); p^{-1}(\partial_0 \sigma), \cdots, p^{-1}(\partial_n \sigma)).$$

Then we call $p: E \to |K|$ a CAT block bundle with fiber F.

Theorem (block bundle theorem)

Let M be a closed TOP manifold of dimension m with free abelian $\pi_1(M)$. If there is a self-covering $M \stackrel{h}{\longrightarrow} M' \stackrel{q}{\longrightarrow} M$. Let $G = \bigcap_{k=1}^{\infty} \operatorname{Im} h_{\#}^k$ with $\pi_1(M)/G = \mathbb{Z}^n$. Then $p \colon M \to \mathbb{T}^n$ is homotopic to a TOP block bundle projection if $m-n \geq 5$.

From block bundle to fiber bundle

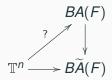
A(F) = the CAT-automorphism group of F

 $\widetilde{A}(F)$ = the block CAT-automorphism group of F.

 $\widetilde{A}(F)$ is a semi-simplicial group, an *n*-simplex in $\widetilde{A}(F)$ is a face-preserving CAT-automorphism

$$\Delta^n \times F \rightarrow \Delta^n \times F$$

The lifting problem



has obstructions $o_i \in H^i(\mathbb{T}^n; \pi_{i-1}\widetilde{A}(F)/A(F))$.

Special case 1: low dimensional base

Theorem

Under the assumption of block bundle theorem. If G = 0 and n = 1 or 2, $m \ge 7$. Then the map $p: M \to \mathbb{T}^n$ is homotopic to a TOP bundle projection.

Proof.

Hatcher's spectral sequence

$$E_{ij}^1 = \pi_j(C(F \times [0,1]^i)) \Rightarrow \pi_{i+j+1}(\widetilde{A}(F)/A(F))$$

where $C(F) = A(F \times I, F \times 0)$ is the concordance group of F.

If $G = \pi_1(F) = 0$ and dim $F \ge 4$, then C(F) is path connected.

Therefore $\pi_1(\widetilde{A}(F)/A(F)) = 0$, $H^2(\mathbb{T}^2; \pi_1(\widetilde{A}(F)/A(F))) = 0$.

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Special case 2: highly connected fiber

Theorem

Under the assumption of block bundle theorem. If G = 0, $\pi_i(M) = 0$ for $2 \le i \le r$, where $m - n \ge r + 4$ and $n \le \min\{2r - 1, r + 4\}$. Then the map $p \colon M \to \mathbb{T}^n$ is homotopic to a TOP bundle projection with r-connected fiber.

Example. If r = 2, then $m - n \ge 6$, $n \le 3$.

Proof.

- The fiber F is r-connected, dim $F = f \ge r + 4$.
- For $j \le n-1$, $\pi_i(\widetilde{A}(F)/A(F), \widetilde{A}_{\partial}(D^f)/A_{\partial}(D^f)) = 0$
- Alexander's trick $\Rightarrow \widetilde{A}_{\partial}(D^f)/A_{\partial}(D^f) \simeq *$.
- Therefore $H^i(\mathbb{T}^n; \pi_{i-1}\widetilde{A}(F)/A(F)) = 0$ for all $i \leq n$.

Special case 3: stabilization

Theorem

Under the assumption of block bundle theorem. Let s = n(n-1)/2. Then

$$M \times \mathbb{T}^s \xrightarrow{\operatorname{pr}_1} M \xrightarrow{p} \mathbb{T}^n$$

is homotopic to a TOP bundle projection.

Proof.

A direct consequence of Burghelea-Lashof-Rothenberg.



Non-fibering examples

Theorem

For any $n \ge 4$, i = 2, 3, there exists a closed TOP manifold M such that

- 1. $M \simeq \mathbb{T}^n \times S^i$ (hence M is homotopy self-covering);
- 2. *M* is not a TOP bundle over \mathbb{T}^n .

If $n \ge 8$, M is a smooth manifold.

Proof.

If $p: M \to \mathbb{T}^n$ is a fiber bundle, then the fiber is S^i , with structure group $\operatorname{Homeo}(S^i) \simeq O(i+1)$, hence is the sphere bundle of a rank i+1 vector bundle. Then $TM = \zeta \oplus p^*T\mathbb{T}^n$, $p_2(M) = 0$.

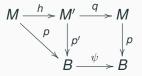
Starting from a degree 1 normal map $f: N \to \mathbb{T}^n$ with non-trivial surgery obstruction. For $f \times \operatorname{id}: N \times S^i \to \mathbb{T}^n \times S^i$ the surgery obstruction is trivial, we get $M \simeq \mathbb{T}^n \times S^i$ with $p_2(M) \neq 0$

Non-abelian fundamental group: fibering problem

A self-covering manifold $M \stackrel{h}{\longrightarrow} M' \stackrel{q}{\longrightarrow} M$ with non-abelian π_1

Question

Is there a bundle projection $p: M \to B$ such that



where B is a closed aspherical manifold, $\psi \colon B \to B$ a covering map such that $\bigcap_{k=1}^{\infty} \operatorname{Im} \psi_{\#}^{k} = 0$.

Define the residue group $G = \bigcap_{k=1}^{\infty} \operatorname{Im} h_{\#}^{k}$, then

$$\pi_1(\mathrm{hfib}(p)) = G = \mathrm{Ker}p_\#$$

Non-abelian fundamental group: a non-fibering example

Necessary conditions:

- 1. *G* is a normal subgroup of $\pi_1(M)$;
- 2. G is finitely presented.

Theorem

Let $BS(2,3) = \langle a,t \mid ta^2t^{-1} = a^3 \rangle$. Then for each $m \geq 5$, there exists a smooth closed m-dimensional manifold M with $\pi_1(M) = BS(2,3)$, such that there is a 5-fold covering map $q \colon M \to M$, with $G = \bigcap_{k=1}^{\infty} \mathrm{Im} q_{\#}^k$. The followings hold

- 1. *G* is not a normal subgroup of $\pi_1(M)$;
- 2. *G* is isomorphic to F_{∞} , hence not finitely generated.

From CW-complexes to manifolds

- X = K(BS(2,3),1) a finite 2-dimensional complex
- there is a 5-fold covering $X' \to X$ such that $\pi_1(X') = \langle a^5, t \rangle < BS(2,3)$
- there is a homeomorphism $h: X \to X'$ with $h_\#(a) = a^{-5}$, $h_\#(t) = t$
- $G = \bigcap_{k=1}^{\infty} \operatorname{Im} h_{\#}^{k} \cong F_{\infty}$ is not a normal subgroup of BS(2,3)

Theorem

Let X be a finite 2-dimensional CW complex, X' be a k-fold covering of X. If there is a homeomorphism $h\colon X\to X'$. Then there exists a compact smooth hypersurface M in \mathbb{R}^6 with $\pi_1(M)=\pi_1(X)$. Let M' be the k-fold covering of M with $\pi_1(M')=\pi_1(X')$, there is a diffeomorphism $\varphi\colon M\to M'$ such that $h_\#=\varphi_\#$.

Finiteness: homotopy

Given a map $p \colon M \to \mathbb{T}^n$, how to show that $\mathrm{hfib}(p) \simeq \overline{M}$ is finitely dominated?

Theorem (Wall)

Let X be a CW-complex with finitely presented $\pi_1(X)$ and $\Lambda = \mathbb{Z}[\pi_1(X)]$ a noetherian ring. Then X is finitely dominated if and only if $H_i(X;\Lambda)$ is a finite Λ -module for all i and the cohomological dimension of X is finite.

Finiteness: algebra

We need to show that $H_i(\overline{M}; \mathbb{Z}[G]) = H_i(\widetilde{M}; \mathbb{Z})$ is a finite $\mathbb{Z}[G]$ -module.

By construction $H_i(\overline{M}; \mathbb{Z}[G]) = H_i(\widetilde{M}; \mathbb{Z})$ is a finite $\mathbb{Z}[\pi_1(M)]$ -module, with $\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[G][\mathbb{Z}^n]$

Theorem

Let R be a commutative noetherian ring, $S = R[t_1^{\pm 1}, \cdots, t_n^{-1}]$. Assume there is a ring monomorphism $\phi \colon S \to S$ with $\phi(R) = R$, and $\phi(t_j) = g_j \prod_{i=1}^n t_i^{a_{ij}}$, where $g_j \in R^\times$ and $A = (a_{ij})_{n \times n}$ satisfies $\det A \neq 0$, $\bigcap_{k=1}^\infty \operatorname{Im} A^k = 0$. If $\mathfrak M$ is a finite S-module and there is a ϕ -twisted S-module isomorphism $\eta \colon \mathfrak M \to \mathfrak M$, i.e., $\eta(sx) = \phi(s)\eta(x)$, then $\mathfrak M$ is a finite R-module.

Thank You