Compactification of homology cells and the complex projective space

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- (Atiyah-Singer 1963)

$$D: \ \Gamma(S^+ \oplus S^-) \xrightarrow{(D^+, D^-)} \Gamma(S^- \oplus S^+), \quad (D\text{: Dirac operator})$$

$$\hat{A}(M) = \dim \ker(D^+) - \dim \ker(D^-).$$

- (Lichnerowicz 1963)

$$D^2 = \nabla^* \nabla + \frac{1}{4} s_g, \quad (s_g\text{: scalar curvature.})$$

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- Their result is indeed inspired by Lusztig and Kosniowski's similar consideration for the χ_y-genus on complex manifolds.

Theorem (Hattori 1978)

Let M be an almost-complex S^1-manifold, $b_1(M) = 0$, and $c_1(M) = kx$ with $k \in \mathbb{Z}_{>0}$ and $x \in H^2(M;\mathbb{Z})$ indivisible. Then

$$\{e^{tx/2}\hat{\mathfrak{A}}(M)\}[M] = 0, \quad \forall\ t \equiv k \pmod 2 \text{ and } |t| < k,$$

where $\hat{\mathfrak{A}}(M) \in H^{4*}(M;\mathbb{Z})$ is the $\hat{A}$-class.

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This result is motivated by

Conjecture (Petrie 1972)

If a cohomology $\mathbb{P}^n$ $(H^*(M;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}))$ admits a circle action, then the total Pontrjagin class $p(M) = (1 + x^2)^{n+1}$.

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This conjecture is equivalent to

$$\{e^{tx/2}\hat{\mathfrak{A}}(M)\}[M] = 0, \quad \forall\ t \equiv n+1 \pmod 2 \text{ and } |t| < n+1.$$

Theorem (Michelsohn 1980)

Let M be a complex manifold such that $c_1(M) = kx$ with $k \in \mathbb{Z}_{>0}$ and $x \in H^2(M;\mathbb{Z})$ indivisible. If M admits a Ricci-positive Kähler *metric, then*

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Remark

The equality characterization has been (implicitly) done by Hirzebruch-Kodaira (1957).

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Example (linear action on $\mathbb{P}^n$)

$$\mathbb{P}^n := \left\{[z_1 : z_2 : \cdots : z_{n+1}] \mid z_i \in \mathbb{C}\right\}$$

and choose $n+1$ pairwise distinct integers $a_1, a_2, \ldots, a_{n+1}$.

$$\begin{aligned} & z \cdot [z_1 : z_2 : \cdots : z_{n+1}] \\ :=& [z^{-a_1} z_1 : z^{-a_2} z_2 : \cdots : z^{-a_{n+1}} z_{n+1}], \quad z \in S^1 \subset \mathbb{C}, \end{aligned}$$

has $n+1$ isolated fixed points

$$P_i = [\underbrace{0 : \cdots : 0}_{i-1}, 1, 0 : \cdots : 0], \quad 1 \leq i \leq n+1.$$

$$T_{P_i}\mathbb{P}^n = \sum_{j \neq i} t^{a_i - a_j} \in \mathbb{Z}[t, t^{-1}] = R(S^1).$$

Definition (Hattori)

- M: almost-complex S^1-manifold with $M^{S^1} = \{P_1, \ldots, P_m\}$.
- L: a line bundle over M to which this S^1-action can be lifted, and $L_{P_i} = t^{a_i}$ w.r.t. some lifting.
- L is called quasi-ample if $c_1^n[M] \neq 0$ and a_i are pairwise distinct.

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Theorem (Hattori 1984)

Let a quasi-ample L be as above, $T_{P_i}M = t^{k_1^{(i)}} + \cdots + t^{k_n^{(i)}}$, and

$$k_1^{(i)} + \cdots + k_n^{(i)} = k_0 a_i + a, \quad \forall\, 1 \leq i \leq m,\ k_0 \in \mathbb{Z}_{\geq 0},\ a \in \mathbb{Z}.$$

Then $k_0 \leq n+1 \leq m$. If $k_0 = n+1 = m$, then M is unitary cobordant to $\mathbb{P}^n$, $c_1^n(L)[M] = 1$, and

$$\{k_1^{(i)}, \dots, k_n^{(i)}\} = \{a_i - a_j \mid j \neq i\}, \quad \forall\, 1 \leq i \leq n+1,$$

i.e., the S^1-actions at P_i are the linear ones on $\mathbb{P}^n$.

Definition

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Question (Problem 27 in Hirzebruch's 1954 Problem List)

Determine all compactifications (M, D) *of* $\mathbb{C}^n$ *with* $b_2(M) = 1$.

$$(b_2(M) = 1 \Longleftrightarrow D \text{ is irreducible} \Longleftarrow D \text{ is smooth.})$$

- (Remmert-van de Ven 1960) When $n = 2$, $(\mathbb{P}^2, \mathbb{P}^1)$ is the only example with $b_2 = 1$.
- (van de Ven 1962, Ramanujam 1971, Kodaira 1972, Morrow 1973) There is a complete classification for all the compactifications of $\mathbb{C}^2$ by relating D to a graph.
- (Brenton-Morrow 1978) When $n = 3$, $(\mathbb{P}^3, \mathbb{P}^2)$ is the only smooth example.
- When D is allowed to be singular, there is a complete but complicated classification for Kähler compactification of $\mathbb{C}^3$, due to Furushima, Nakayama, Peternell and Schneider.
- (van de Ven 1962, Fujita 1980) When $n \leq 6$, $(\mathbb{P}^n, \mathbb{P}^{n-1})$ are the only smooth Kähler example.

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Theorem (Chi Li-Zhengyi Zhou 2024)

The only smooth Kähler compactification of $\mathbb{C}^n$ is $(\mathbb{P}^n, \mathbb{P}^{n-1})$.

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- When $n \geq 3$, no essential result on complex structure.
- (Hirzebruch 1954)
  the uniqueness of complex structure on $\mathbb{P}^3$ would imply the nonexistence of complex structure on S^6.

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Theorem (Fujita 1980, Lanteri-Struppa 1983, Wilson 1986, Libgober-Wood 1990, Debarre 2015)

M is compact Kähler with $H^*(M;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$.

1. *If $n = 3, 5$, then $M = \mathbb{P}^n$;*
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- Key point "$b_i(M) = b_i(\mathbb{P}^n) \Longrightarrow c_1 c_{n-1}[M] = \frac{1}{2}n(n+1)^2$" (Libgober-Wood 1990)
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- No known examples of quotients of even dimensional unit balls $\mathbb{B}^{2k}$ with the same integral cohomology ring as $\mathbb{P}^{2k}$.
- Let Q^n be the quadratic hypersurface in $\mathbb{P}^{n+1}$. Then

$$H_*(Q^{2k+1};\mathbb{Z}) \cong H_*(\mathbb{P}^{2k+1};\mathbb{Z}).$$

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Let M be Kähler and D a smooth hypersurface on it. If

$$H_k(D;\mathbb{Z}) \xrightarrow{\cong} H_k(M;\mathbb{Z}), \quad 0 \leq k \leq 2(n-1),$$

which is call $M \setminus D$ being a homology cell. Then $(M, D) = (\mathbb{P}^n, \mathbb{P}^{n-1})$.

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Let U be contractible and (M, D) a Kähler smooth compactification. Then $(M, D) = (\mathbb{P}^n, \mathbb{P}^{n-1})$, and hence $U \cong \mathbb{C}^n$.

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Let U be contractible and (M, D) a Kähler smooth compactification. Then $(M, D) = (\mathbb{P}^n, \mathbb{P}^{n-1})$, and hence $U \cong \mathbb{C}^n$.

- (Fujita) "$C_n \Longrightarrow B_n \Longrightarrow A_n$" and C_n true when $n \leq 5$.

Theorem (Peternell-L. 2025)

Let (M, D) be a Kähler smooth compactification of a homology cell $M \setminus D$.

1. *If $n \not\equiv 3 \pmod 4$, $(M, D) = (\mathbb{P}^n, \mathbb{P}^{n-1})$.*
2. *If $n \equiv 3 \pmod 4$, either $(M, D) = (\mathbb{P}^n, \mathbb{P}^{n-1})$, or M^n and D^{n-1} are Fano manifolds with Fano indices $\frac{1}{2}(n+1)$ and $\frac{1}{2}(n-1)$ respectively.*

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- $M \setminus D$: contractible$\Longrightarrow$ homologically trivial $\left(\widetilde{H}_i(\cdot) = 0\right)\Longrightarrow$ being homology cell.

Theorem (Peternell-L. 2025)

Let (M, D) be a Kähler smooth compactification of a homology cell $M \setminus D$.

1. *If $n \not\equiv 3 \pmod 4$, $(M, D) = (\mathbb{P}^n, \mathbb{P}^{n-1})$.*
2. *If $n \equiv 3 \pmod 4$, either $(M, D) = (\mathbb{P}^n, \mathbb{P}^{n-1})$, or M^n and D^{n-1} are Fano manifolds with Fano indices $\frac{1}{2}(n+1)$ and $\frac{1}{2}(n-1)$ respectively.*

- $M \setminus D$: contractible$\Longrightarrow$ homologically trivial $\left(\widetilde{H}_i(\cdot) = 0\right)$$\Longrightarrow$ being homology cell.

Corollary (Peternell-L. 2025)

Fujita's conjectures B_n, and hence A_n are true provided that $n \not\equiv 3 \pmod 4$.

Key points in the proof.

1. M and D are cohomolgy $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$.
2. $$\text{“}c_1 c_{n-1}[M] = \frac{1}{2}n(n+1)^2\text{”} + \text{“}c_1 c_{n-2}[D] = \frac{1}{2}(n-1)n^2\text{”}$$

   yields that

   $$\big(c_1(M), c_1(D)\big) = (n+1, n) \text{ or } (\frac{1}{2}(n+1), \frac{1}{2}(n-1)).$$

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Question

Can we remove the case of $\big(c_1(M),c_1(D)\big)=\big(\frac{1}{2}(n+1),\frac{1}{2}(n-1)\big)$ *when* “$n\equiv 3\pmod 4$”?

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Example (linear action on $(\mathbb{P}^n, \mathbb{P}^{n-1})$)

$$\mathbb{P}^{n-1} := \left\{[z_1 : z_2 : \cdots : z_{n+1}] \mid z_{n+1} = 0\right\} \subset \mathbb{P}^n$$

$$z \cdot [z_1 : z_2 : \cdots : z_{n+1}]$$
$$:=[z^{-a_1} z_1 : z^{-a_2} z_2 : \cdots : z^{-a_{n+1}} z_{n+1}], \quad z \in S^1 \subset \mathbb{C},$$

has $n+1$ fixed points

$$P_i = [\underbrace{0 : \cdots : 0}_{i-1}, 1, 0 : \cdots : 0], \quad 1 \leq i \leq n+1,$$

and is S^1-invariant on $\mathbb{P}^{n-1}$ with fixed points $P_1, \ldots, P_n$.

$$\begin{cases} T_{P_i}\mathbb{P}^n = \sum_{1\leq j\leq n+1, j\neq i} t^{a_i - a_j} \in R(S^1), \quad 1 \leq i \leq n+1, \\ T_{P_i}\mathbb{P}^{n-1} = \sum_{1\leq j\leq n, j\neq i} t^{a_i - a_j} \in R(S^1), \quad 1 \leq i \leq n. \end{cases}$$

Theorem (L. 2025)

Let M be a symplectic manifold admitting a Hamiltonian circle action with isolated fixed points. If M contains an S^1-invariant symplectic hypersurface D such that $M \setminus D$ is a homology cell. Then

- *M and D are both homotopy complex projective spaces.*
- *When $n \not\equiv 3 \pmod 4$, M and D have standard Chern classes and the S^1-representations on their fixed points are the same as those arising from the linear action on $(\mathbb{P}^n, \mathbb{P}^{n-1})$ for some pairwise distinct integers $a_1, \ldots, a_{n+1}$.*
- *When $n \equiv 3 \pmod 4$, the conclusions above are still true provided that*

$$\big(c_1(M), c_1(D)\big) \neq \big(\frac{1}{2}(n+1), \frac{1}{2}(n-1)\big).$$

谢谢!