Topological complexity of enumerative problems in algebraic geometry

Xing Gu, joint work with Weiyan Chen

Westlake University

guxing@westlake.edu.cn

South China Normal University

2025-11-09

Overview

Backgrounds

Backgrounds

Enumerative problems in algebraic geometry (over $\mathbb{C}$):

1. a generic degree d-polynomial has d roots;
2. a generic quartic plane curve has 24 inflection points;
3. a smooth quartic plane curve has 28 bitangent lines;
4. a smooth cubic curve contains 27 lines;
5. ...

Backgrounds

Enumerative problems in algebraic geometry (over $\mathbb{C}$):

1. a generic degree d-polynomial has d roots;
2. a generic quartic plane curve has 24 inflection points;
3. a smooth quartic plane curve has 28 bitangent lines;
4. a smooth cubic curve contains 27 lines;
5. ...

A positive integer, which we call the *topological complexity* may be assigned to each of the enumerative problems in algebraic geometry.

Backgrounds

In 1987, S. Smale [5] considered the topological complexity of finding the d roots of a generic degree-d polynomial:

Poly(d)

Given a generic complex polynomial of degree d, leading coefficients 1 and $\epsilon > 0$, Find all roots of f within ϵ.

The topological complexity of enumerative problems

The topological complexity of enumerative problems

Following Smale, by an *algorithm* we mean a finite rooted tree consisting of a root for the input, leaves for the output, and internal nodes of the following two types:

computation nodes and *branching nodes* .

The topological complexity of enumerative problems

Following Smale, by an *algorithm* we mean a finite rooted tree consisting of a root for the input, leaves for the output, and internal nodes of the following two types:

computation nodes and *branching nodes* .

The *topological complexity of enumerative problems* of an algorithm is the number of branching nodes in the tree.

Following Smale, by an *algorithm* we mean a finite rooted tree consisting of a root for the input, leaves for the output, and internal nodes of the following two types:

computation nodes and *branching nodes* .

The *topological complexity of enumerative problems* of an algorithm is the number of branching nodes in the tree.
The topological complexity of a problem P is the minimum of the topological complexity of any algorithm solving P.

The topological complexity of enumerative problems

Following Smale, by an *algorithm* we mean a finite rooted tree consisting of a root for the input, leaves for the output, and internal nodes of the following two types:

computation nodes *and* *branching nodes* .

The *topological complexity of enumerative problems* of an algorithm is the number of branching nodes in the tree.
The topological complexity of a problem P is the minimum of the topological complexity of any algorithm solving P.
Smale [5] first proved a lower bound for the topological complexity of the problem of finding roots of a polynomial $f(z) = 0$.

The topological complexity of enumerative problems

Following Smale, by an *algorithm* we mean a finite rooted tree consisting of a root for the input, leaves for the output, and internal nodes of the following two types:

computation nodes *and* *branching nodes* .

The *topological complexity of enumerative problems* of an algorithm is the number of branching nodes in the tree.
The topological complexity of a problem P is the minimum of the topological complexity of any algorithm solving P.
Smale [5] first proved a lower bound for the topological complexity of the problem of finding roots of a polynomial $f(z) = 0$. Smale's lower bounds were later improved by Vassiliev [6], De Concini-Procesi-Salvetti [3], and Arone [1].

The topological complexity of enumerative problems

Similar questions may be asked about solutions to enumerative problems for high-dimensional objects such as curves and surfaces, which is mostly unknown.

The topological complexity of enumerative problems

Similar questions may be asked about solutions to enumerative problems for high-dimensional objects such as curves and surfaces, which is mostly unknown. A first step in this direction is the work [2] by Weiyan Chen and Zheyan Wan, concerning the topological complexity of finding inflection points on cubic plane curves.

The topological complexity of enumerative problems

Similar questions may be asked about solutions to enumerative problems for high-dimensional objects such as curves and surfaces, which is mostly unknown. A first step in this direction is the work [2] by Weiyan Chen and Zheyan Wan, concerning the topological complexity of finding inflection points on cubic plane curves. In this talk, we consider **the lower bounds of the topological complexity** of the following problems:

- **Line**(ϵ): Given any cubic surface defined by a homogeneous polynomial $F(x, y, z, w)$ of degree 3, find all of its 27 lines $(l_1, \cdots, l_{27})$ within ϵ.
- **Bitangent**(ϵ): Given any quartic curve defined by a homogeneous polynomial $F(x, y, z)$ of degree 4, find all of its 28 bitangent lines $(l_1, \cdots, l_{28})$ within ϵ.
- **Flex**(ϵ): Given any quartic curve defined by a homogeneous polynomial $F(x, y, z)$ of degree 4, find all of its 24 inflection points $(p_1, \cdots, p_{28})$ within ϵ.

The topological complexity of enumerative problems

Theorem

When ϵ is sufficiently small, we have

1. *the topological complexity of the problem* $\textbf{Line}(\epsilon)$ *is at least 15,*
2. *the topological complexity of the problem* $\textbf{Bitangent}(\epsilon)$ *is at least 8,*
3. *the topological complexity of the problem* $\textbf{Flex}(\epsilon)$ *is at least 8.*

Boader context

A General Question

How complex it is to find the roots of a polynomial?

Boader context

A General Question

How complex it is to find the roots of a polynomial?

The answer depends on what "find" means.

- find = express in radicals
  complexity $\sim$ (solvability of) the Galois group

Boader context

A General Question

How complex it is to find the roots of a polynomial?

The answer depends on what “find” means.

- find $=$ express in radicals
  complexity $\sim$ (solvability of) the Galois group
- find $=$ express in algebraic functions (Hilbert's 13th problem)
  complexity $\sim$ resolvent degrees (Brauer 1975)

Boader context

A General Question

How complex it is to find the roots of a polynomial?

The answer depends on what "find" means.

- find = express in radicals
  complexity $\sim$ (solvability of) the Galois group
- find = express in algebraic functions (Hilbert's 13th problem)
  complexity $\sim$ resolvent degrees (Brauer 1975)
- find = approximate using an algorithm
  complexity $\sim$ topological complexity (Smale 1987)

The topological construction associated to the enumerative problems

There are various parameter spaces associated to the three enumerative problems. For instance, for the problem **Line**(ϵ), consider

$$B_{\text{line}} := \{\text{nonsingular homogeneous cubic polynomials } F(x,y,z,w)\}/\mathbb{C}^\times$$
$$E_{\text{line}} := \{(F, l_1, \cdots, l_{27}) : \ l_i\text{'s are the 27 lines on the cubic surface } F \in B_{\text{line}}\}$$

The space B_{line} is an open submanifold of $\mathbb{C}P^{19}$ consisting of all homogeneous cubic polynomials in four variables.

The topological construction associated to the enumerative problems

There are various parameter spaces associated to the three enumerative problems. For instance, for the problem **Line**(ϵ), consider

$$B_{\text{line}} := \{\text{nonsingular homogeneous cubic polynomials } F(x,y,z,w)\}/\mathbb{C}^\times$$
$$E_{\text{line}} := \{(F, l_1, \cdots, l_{27}) : \ l_i\text{'s are the 27 lines on the cubic surface } F \in B_{\text{line}}\}$$

The space B_{line} is an open submanifold of $\mathbb{C}P^{19}$ consisting of all homogeneous cubic polynomials in four variables.
The projection $E_{\text{line}} \to B_{\text{line}}$ given by $(F, l_1, \cdots, l_{27}) \mapsto F$ is a normal S_{27}-cover, where S_{27} acts on E_{line} by permuting the ordering of the 27 lines.

The topological construction associated to the enumerative problems

There are various parameter spaces associated to the three enumerative problems. For instance, for the problem **Line**(ϵ), consider

$$B_{\text{line}} := \{\text{nonsingular homogeneous cubic polynomials } F(x, y, z, w)\}/\mathbb{C}^{\times}$$
$$E_{\text{line}} := \{(F, l_1, \cdots, l_{27}) : \ l_i\text{'s are the 27 lines on the cubic surface } F \in B_{\text{line}}\}$$

The space B_{line} is an open submanifold of $\mathbb{C}P^{19}$ consisting of all homogeneous cubic polynomials in four variables.

The projection $E_{\text{line}} \to B_{\text{line}}$ given by $(F, l_1, \cdots, l_{27}) \mapsto F$ is a normal S_{27}-cover, where S_{27} acts on E_{line} by permuting the ordering of the 27 lines.

Similarly, we have a S_{28}-cover $E_{\text{btg}} \to B_{\text{btg}}$ associated to the problem **Bitangent**(ϵ) and another S_{28}-cover $E_{\text{flex}} \to B_{\text{flex}}$ associated to the problem **Flex**(ϵ).

The topological construction associated to the enumerative problems

Definition ([4])

The *Schwarz genus* of a covering $E \to B$, denoted by $g(E \to B)$ or simply $g(E)$, is the minimum size of an open cover of B consisting of open sets such that there exists a continuous section of the covering map $E \to B$ over each open set.

The topological construction associated to the enumerative problems

Definition ([4])

The *Schwarz genus* of a covering $E \to B$, denoted by $g(E \to B)$ or simply $g(E)$, is the minimum size of an open cover of B consisting of open sets such that there exists a continuous section of the covering map $E \to B$ over each open set.

Theorem

The topological complexity of the problem $\mathbf{Line}(\epsilon)$ *is* $g(E_{line} \to Bl) - 1$. *Similar equations holds for the problems* $\mathbf{Bitangent}(\epsilon)$ *and* $\mathbf{Flex}(\epsilon)$.

The topological construction associated to the enumerative problems

Proposition ([4], p.71)

Let $i^*E \to B'$ denote the pullback of a cover $E \to B$ along a continuous map $i : B' \to B$. Then $g(i^*E \to B') \leq g(E \to B)$.

The topological construction associated to the enumerative problems

Proposition ([4], p.71)

Let $i^*E \to B'$ denote the pullback of a cover $E \to B$ along a continuous map $i : B' \to B$. Then $g(i^*E \to B') \leq g(E \to B)$.

Proposition ([4], p.76)

If B is a CW complex of dimension d, then $g(E \to B) \leq d + 1$.

The topological construction associated to the enumerative problems

Proposition ([4], p.71)

Let $i^*E \to B'$ denote the pullback of a cover $E \to B$ along a continuous map $i : B' \to B$. Then $g(i^*E \to B') \leq g(E \to B)$.

Proposition ([4], p.76)

If B is a CW complex of dimension d, then $g(E \to B) \leq d + 1$.

Proposition (Disconnected covers)

Consider a cover $E \to B$ with path-components $E = \bigcup_{i \in I} E_i$ where each $E_i \to B$ is also a cover. Suppose that there exists an $m \in I$ such that for any $i \in I$, there exists morphism of coverings $E_i \to E_m$. Then $g(E) = g(E_m)$.

The topological construction associated to the enumerative problems

Proposition ([4], p.98)

Suppose that $E \to B$ is a a principal Γ-bundle with a classifying map $cl : B \to B\Gamma$ where $B\Gamma$ is the classifying space of Γ. Then we have

$$g(E \to B) \geq \min\{k : \mathrm{H}^i(B\Gamma; A) \xrightarrow{cl^*} \mathrm{H}^i(B; A) \text{ is zero for any } i \geq k\}$$

for any Γ-module A. The integer on the right hand side of the inequality above is called the *homological A-genus* of the cover $E \to B$.

The topological construction associated to the enumerative problems

Summary

Let $TC(P)$ be the topological complexity of an enumerative problem P. Let $E_P \to B_P$ be the covering space associated to the problem P. Let $E \to B$ be a pullback of $E_P \to B_P$. Let $g(-)$ and $g_A(-)$ be the Schwarz genus and A-genus, respectively. Then we have

The topological construction associated to the enumerative problems

Summary

Let $TC(P)$ be the topological complexity of an enumerative problem P. Let $E_P \to B_P$ be the covering space associated to the problem P. Let $E \to B$ be a pullback of $E_P \to B_P$. Let $g(-)$ and $g_A(-)$ be the Schwarz genus and A-genus, respectively. Then we have

$$TC(P) = g(E_P \to B_P) - 1 \geq g(E \to B) - 1 \geq g_A(E \to B) - 1.$$

The topological construction associated to the enumerative problems

Summary

Let $TC(P)$ be the topological complexity of an enumerative problem P. Let $E_P \to B_P$ be the covering space associated to the problem P. Let $E \to B$ be a pullback of $E_P \to B_P$. Let $g(-)$ and $g_A(-)$ be the Schwarz genus and A-genus, respectively. Then we have

$$TC(P) = g(E_P \to B_P) - 1 \geq g(E \to B) - 1 \geq g_A(E \to B) - 1.$$

Next, we look for a pullback $E \to B$ such that we can calculate $g_A(E \to B)$.

Finding lower bounds by pullbacks

Finding lower bounds by pullbacks

We focus on the problem **Line**(ϵ), i.e., the problem of finding (within ϵ) the 27 lines on a smooth cubic surface.

Finding lower bounds by pullbacks

We focus on the problem **Line**(ϵ), i.e., the problem of finding (within ϵ) the 27 lines on a smooth cubic surface.Recall

$$B_{\text{line}} := \{\text{nonsingular homogeneous cubic polynomials } F(x, y, z, w)\}/\mathbb{C}^{\times}$$
$$E_{\text{line}} := \{(F, l_1, \cdots, l_{27}) : \ l_i\text{'s are the 27 lines on the cubic surface } F \in B_{\text{line}}\}$$

The space B_{line} is an open submanifold of $\mathbb{C}P^{19}$ consisting of all homogeneous cubic polynomials in four variables.

Finding lower bounds by pullbacks

We focus on the problem **Line**(ϵ), i.e., the problem of finding (within ϵ) the 27 lines on a smooth cubic surface. Recall

$$B_{\text{line}} := \{\text{nonsingular homogeneous cubic polynomials } F(x, y, z, w)\}/\mathbb{C}^\times$$
$$E_{\text{line}} := \{(F, l_1, \cdots, l_{27}) : \ l_i\text{'s are the 27 lines on the cubic surface } F \in B_{\text{line}}\}$$

The space B_{line} is an open submanifold of $\mathbb{C}P^{19}$ consisting of all homogeneous cubic polynomials in four variables. The projective unitary group $\mathrm{PU}(4)$ acts on B_{line} by acting on the variables (x, y, z, w).

Finding lower bounds by pullbacks

We consider the Fermat cubic surface

$$F(x,y,z,w) = x^3 + y^3 + z^3 + w^3.$$

Consider the following subgroup $K \leq \mathrm{PU}_4$ preserving the Fermat cubic surface F:

$$K := \left\langle \begin{bmatrix} e^{2\pi i/3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle \cong (\mathbb{Z}/3\mathbb{Z})^3, \tag{1}$$

Finding lower bounds by pullbacks

We consider the Fermat cubic surface

$$F(x,y,z,w) = x^3 + y^3 + z^3 + w^3.$$

Consider the following subgroup $K \leq \mathrm{PU}_4$ preserving the Fermat cubic surface F:

$$K := \left\langle \begin{bmatrix} e^{2\pi i/3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle \cong (\mathbb{Z}/3\mathbb{Z})^3, \tag{1}$$

which leaves $F(x,y,z,w)$ fixed.

Finding lower bounds by pullbacks

Lemma

The action of K on the set of 27 lines on the Fermat cubic surface is faithful. In other words, the induced group homomorphism $K \to S_{27}$ is injective.

Finding lower bounds by pullbacks

Lemma

The action of K on the set of 27 lines on the Fermat cubic surface is faithful. In other words, the induced group homomorphism $K \to S_{27}$ is injective.

Therefore, we have an injective map

$$\eta\colon PU(4)/K \to B_{\text{line}}, \ [\sigma] \mapsto \sigma F.$$

Finding lower bounds by pullbacks

Proposition (Chen-G., 2024)

The total space of the pullback of η is homeomorphic to a disjoint union of copies of $\mathrm{PU}(4)$:

$$\eta^* E_{\text{line}} \cong \bigcup_{S_{27}/K} \mathrm{PU}_4$$

where the components are in bijection with K-cosets in S_{27}. In particular, we have

$$g(\eta^* E_{\text{line}} \to \mathrm{PU}(4)/K) = g(\mathrm{PU}(4) \phi PU(4)/K),$$

where ϕ is the quotient map, and a principal K-bundle.

Finding lower bounds by pullbacks

Theorem (Chen-G, 2025)

The homomorphism

$$\phi^*\colon H^*(\mathrm{PU}(4)/K;\mathbb{F}_3) \to H^*(PU(4);\mathbb{F}_3)$$

is surjective.

Finding lower bounds by pullbacks

Theorem (Chen-G, 2025)

The homomorphism

$$\phi^*\colon H^*(\mathrm{PU}(4)/K;\mathbb{F}_3) \to H^*(PU(4);\mathbb{F}_3)$$

is surjective. In particular, we have $g_{\mathbb{F}_3}(E_{line} \to B_{line}) = 15$.

Finding lower bounds by pullbacks

Theorem (Chen-G, 2025)

The homomorphism

$$\phi^*\colon H^*(\mathrm{PU}(4)/K;\mathbb{F}_3) \to H^*(PU(4);\mathbb{F}_3)$$

is surjective. In particular, we have $g_{\mathbb{F}_3}(E_{line} \to B_{line}) = 15$.

Corollary

$g(E_{line} \to B_{line}) \geq 16$.

Finding lower bounds by pullbacks

Theorem (Chen-G, 2025)

The homomorphism

$$\phi^*\colon H^*(\mathrm{PU}(4)/K;\mathbb{F}_3) \to H^*(PU(4);\mathbb{F}_3)$$

is surjective. In particular, we have $g_{\mathbb{F}_3}(E_{line} \to B_{line}) = 15$.

Corollary

$g(E_{line} \to B_{line}) \geq 16$.

Theorem (Chen-G., 2025)

The topological complexity of the problem **Line**(ϵ) *is no less than* 15.

The cohomology of $\mathrm{PU}(4)/K$

The cohomology of $\mathrm{PU}(4)/K$

We have a short exact sequence of Lie groups

$$1 \to K \to \mathrm{PU}(4) \to \mathrm{PU}(4)/K \to 1,$$

which induces a homotopy fiber sequence

$$\mathrm{PU}(4)/K \xrightarrow{cl} BK \xrightarrow{\varphi} B\mathrm{PU}(4)$$

The cohomology of $\mathrm{PU}(4)/K$

We have a short exact sequence of Lie groups

$$1 \to K \to \mathrm{PU}(4) \to \mathrm{PU}(4)/K \to 1,$$

which induces a homotopy fiber sequence

$$\mathrm{PU}(4)/K \xrightarrow{cl} BK \xrightarrow{\varphi} B\mathrm{PU}(4)$$

where cl is the classifying map of the principal K-bundle $\mathrm{PU}(4) \to \mathrm{PU}(4)/K$,

The cohomology of $\mathrm{PU}(4)/K$

We have a short exact sequence of Lie groups

$$1 \to K \to \mathrm{PU}(4) \to \mathrm{PU}(4)/K \to 1,$$

which induces a homotopy fiber sequence

$$\mathrm{PU}(4)/K \xrightarrow{cl} BK \xrightarrow{\varphi} B\mathrm{PU}(4)$$

where cl is the classifying map of the principal K-bundle $\mathrm{PU}(4) \to \mathrm{PU}(4)/K$,and φ is the map induced by the inclusion $K \to \mathrm{PU}(4)$.

The cohomology of $\mathrm{PU}(4)/K$

Consider the map $BK \xrightarrow{\varphi} B\mathrm{PU}(4)$. We have

$$H^*(B\mathrm{PU}(4);\mathbb{F}_3) \cong \mathbb{F}_3[\epsilon_4, \epsilon_6, \epsilon_8], \ \deg epl_i = i,$$

and

$$H^*(BK;\mathbb{F}_3) \cong \mathbb{F}_3[\xi_1, \xi_2, \xi_3] \otimes \Lambda_{\mathbb{F}_3}[u_1, u_2, u_3],$$

where $\deg \xi_i = 2$, $\deg u_i = 1$.

$$\begin{aligned}
\varphi^*(\epsilon_4) &= 3\sigma_1^2 - 8\sigma_2, \\
\varphi^*(\epsilon_6) &= \sigma_1^3 - 4\sigma_1\sigma_2 + 8\sigma_3, \\
\varphi^*(\epsilon_8) &= 3\sigma_1^4 - 16\sigma_1^2 c_2 + 64\sigma_1\sigma_3 - 256\sigma_4,
\end{aligned}$$

where $\sigma_1 = \xi_1 + \xi_2 + \xi_3$, $\sigma_2 = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1$, and $\sigma_3 = \xi_1\xi_2\xi_3$.

The cohomology of $\mathrm{PU}(4)/K$

The homomorphism $\varphi^*\colon H^*(B\mathrm{PU}(4);\mathbb{F}_3) \to H^*(BK;\mathbb{F}_3)$ makes $H^*(BK;\mathbb{F}_3)$ into a $H^*(B\mathrm{PU}(4);\mathbb{F}_3)$-module.

The cohomology of $\mathrm{PU}(4)/K$

The homomorphism $\varphi^*\colon H^*(B\mathrm{PU}(4);\mathbb{F}_3) \to H^*(BK;\mathbb{F}_3)$ makes $H^*(BK;\mathbb{F}_3)$ into a $H^*(B\mathrm{PU}(4);\mathbb{F}_3)$-module.

The Eilenberg-Moore spectral sequence for $\mathrm{PU}(4)/K \xrightarrow{cl} BK \xrightarrow{\varphi} B\mathrm{PU}(4)$ is of the from

$$E_2 = \mathrm{Tor}_{H^*((B\mathrm{PU}(4);\mathbb{F}_3)}(\mathbb{F}_3, H^*(BK;\mathbb{F}_3)) \Rightarrow H^*(PU(4)/K;\mathbb{F}_3).$$

The cohomology of $\mathrm{PU}(4)/K$

The homomorphism $\varphi^*\colon H^*(B\mathrm{PU}(4);\mathbb{F}_3) \to H^*(BK;\mathbb{F}_3)$ makes $H^*(BK;\mathbb{F}_3)$ into a $H^*(B\mathrm{PU}(4);\mathbb{F}_3)$-module.

The Eilenberg-Moore spectral sequence for $\mathrm{PU}(4)/K \xrightarrow{cl} BK \xrightarrow{\varphi} B\mathrm{PU}(4)$ is of the from

$$E_2 = \mathrm{Tor}_{H^*((B\mathrm{PU}(4);\mathbb{F}_3)}(\mathbb{F}_3, H^*(BK;\mathbb{F}_3)) \Rightarrow H^*(PU(4)/K;\mathbb{F}_3).$$

Using the Koszul resolution of $\mathbb{F}_3$, we compute the E_2-page, concentrated in homological degree 0, and collapses at the E_2-page.

The cohomology of $\mathrm{PU}(4)/K$

The homomorphism $\varphi^*\colon H^*(B\mathrm{PU}(4);\mathbb{F}_3) \to H^*(BK;\mathbb{F}_3)$ makes $H^*(BK;\mathbb{F}_3)$ into a $H^*(B\mathrm{PU}(4);\mathbb{F}_3)$-module.

The Eilenberg-Moore spectral sequence for $\mathrm{PU}(4)/K \xrightarrow{cl} BK \xrightarrow{\varphi} B\mathrm{PU}(4)$ is of the from

$$E_2 = \mathrm{Tor}_{H^*((B\mathrm{PU}(4);\mathbb{F}_3)}(\mathbb{F}_3, H^*(BK;\mathbb{F}_3)) \Rightarrow H^*(PU(4)/K;\mathbb{F}_3).$$

Using the Koszul resolution of $\mathbb{F}_3$, we compute the E_2-page, concentrated in homological degree 0, and collapses at the E_2-page.

The lower bounds of the topological complexity of **Bitangent**(ϵ) and **Flex**(ϵ) are obtained in similar ways.

Further Questions

- Better approximations of the lower bounds of the aforementioned problems. This may involve understanding the cohomology of B_{line};

Further Questions

- Better approximations of the lower bounds of the aforementioned problems. This may involve understanding the cohomology of B_{line};
- enumerative Problems in higher dimensions;

Further Questions

- Better approximations of the lower bounds of the aforementioned problems. This may involve understanding the cohomology of B_{line};
- enumerative Problems in higher dimensions;
- geometric interpretations of the cohomology classes of $B\mathrm{PU}(n)$.

Thank You!

G. Arone.
A note on the homology of Σ_n, the Schwartz genus, and solving polynomial equations.
Contemporary Mathematics, 399:1, 2006.

W. Chen and Z. Wan.
Topological complexity of finding flex points on cubic plane curves.
arXiv:2306.17303.

C. De Concini, C. Procesi, and M. Salvetti.
On the equation of degree 6.
Commentarii Mathematici Helvetici, 79:605–617, 2004.

A. Schwarz.
The genus of a fiber space.
Amer. Math. Soc. Transl., 2, 1966.

S. Smale.
On the topology of algorithms. I.
J. Complexity, 3(2):81–89, 1987.

V. A. Vassiliev.
Cohomology of braid groups and the complexity of algorithms.
Funktsional. Anal. i Prilozhen., 22(3):15–24, 1989.