Homology description of equivariant bordism group $\mathcal{Z}_{n+1}(\mathbb{Z}_2^n)$ of $(n+1)$-dim $\mathbb{Z}_2^n$-manifolds with isolated fixed points

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Joint work with Professor Zhi Lü

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Contents

Background

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be the equivariant bordism group of m-dimensional closed manifolds with effective $\mathbb{Z}_2^n$-actions fixing isolated points. Denote by $\mathcal{Z}_*(\mathbb{Z}_2^n) = \sum_m \mathcal{Z}_m(\mathbb{Z}_2^n)$ a graded algebra.

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- Stong(1970) [2]: the homomorphism

$$\begin{aligned} \phi_* : \mathcal{Z}_*(\mathbb{Z}_2^n) &\to \mathcal{R}_*(\mathbb{Z}_2^n) = \mathbb{Z}_2[\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)] \\ [M] &\mapsto \sum_{x \in M^{\mathbb{Z}_2^n}} \tau_x M \end{aligned}$$

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  is injective.
- Since then, the ring structure of $\mathcal{Z}_*(\mathbb{Z}_2^n)$, even the group $\mathcal{Z}_m(\mathbb{Z}_2^n)$, has remained undetermined for $n \geq 3$.

Focus on the Group $\mathcal{Z}_m(\mathbb{Z}_2^n)$

$$\phi_m : \mathcal{Z}_m(\mathbb{Z}_2^n) \hookrightarrow \mathcal{R}_m(\mathbb{Z}_2^n)$$

Three basic problems on $\mathcal{Z}_m(\mathbb{Z}_2^n)$

(**P1**) *Characterize the image of ϕ_m*, i.e., determine which polynomials in $\mathcal{R}_m(\mathbb{Z}_2^n)$ arise as tangent representations at fixed points of a $\mathbb{Z}_2^n$-manifolds.

(**P2**) *Determine the dimension of $\mathcal{Z}_m(\mathbb{Z}_2^n)$* as a vector space over $\mathbb{Z}_2$ for every m and n. Note that $\mathcal{Z}_m(\mathbb{Z}_2^n)$ is trivial for $0 < m < n$ by the effectiveness of $\mathbb{Z}_2^n$-actions.

(**P3**) Which specific types of $\mathbb{Z}_2^n$-manifolds can be used as *preferred representatives* in equivariant bordism classes of $\mathcal{Z}_m(\mathbb{Z}_2^n)$?

Progress on $\mathcal{Z}_m(\mathbb{Z}_2^n)$

- Lü(2009): $(3,3)$:
  - (P2): $\dim \mathcal{Z}_3(\mathbb{Z}_2^3) = 13$,
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  - (P1): a faithful polynomial $f \in$ Im ϕ_n iff $d(f^*) = 0$, where f^* is the dual of f.
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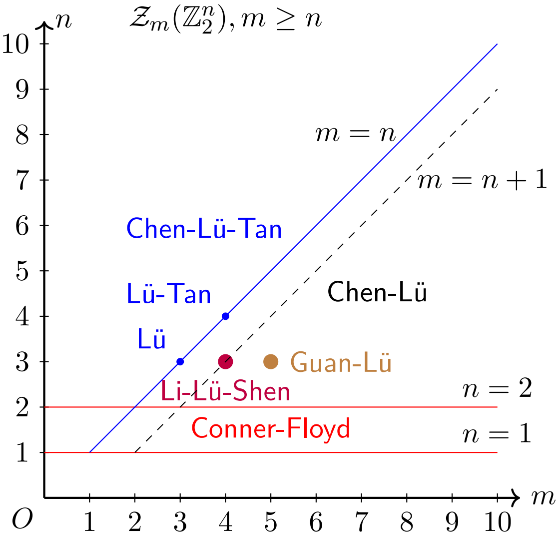
- Guan-Lü(2025): $(3,5)$: (P2): $\dim \mathcal{Z}_5(\mathbb{Z}_2^3) = 77$,
  - (P3): basis: projectivization of real vector bundles.

Main theorem

Progress overview: a visual summary



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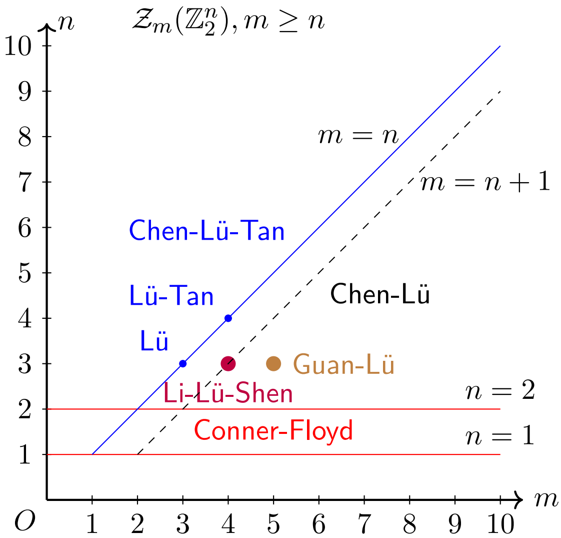


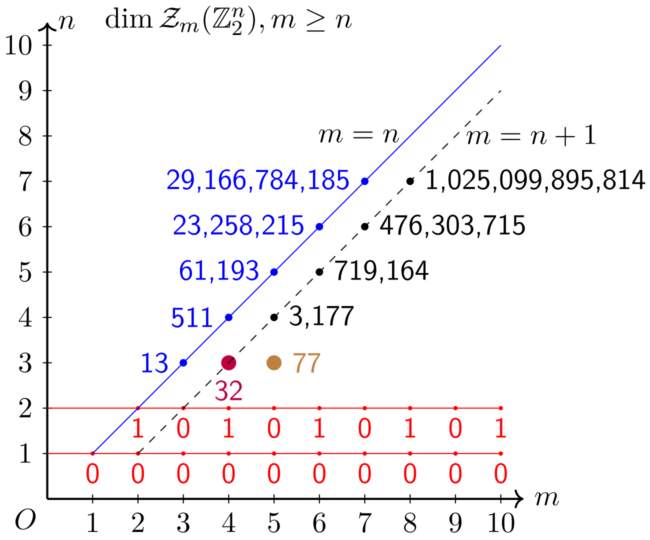
Main Result

For $\mathcal{Z}_{n+1}(\mathbb{Z}_2^n)$, we prove that

- $\mathcal{Z}_{n+1}(\mathbb{Z}_2^n) \cong H_{n-2}(\mathfrak{B})$, where $\mathfrak{B}$ is a chain complex constructed from $X(\mathbb{Z}_2^n)$; and
- a formula for $\dim \mathcal{Z}_{n+1}(\mathbb{Z}_2^n)$.

n	$\dim \mathcal{Z}_{n+1}(\mathbb{Z}_2^n)$
$1, 2$	0
3	32
4	$3,177$
5	$719,164$
6	$476,303,715$
7	$1,025,099,895,814$

Dimensions of equivariant bordism groups



Ideas & Proof

Idea for $\mathcal{Z}_{n+1}(\mathbb{Z}_2^n)$

Inspired by

Effectiveness of dualization in the work of $\mathcal{Z}_n(\mathbb{Z}_2^n)$,

we translate

the LLS detection method

into a simpler dual formulation

$$\sum_{\tau\in\mathcal{A}} \tau \in \mathrm{Im}\ \phi_{n+1} \Longleftrightarrow \partial_{n-2} \sum_{\tau\in\mathcal{A}} D(\tau) = 0.$$

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More specifically, we prove that the sequence

$$0 \longrightarrow \mathcal{Z}_{n+1}(\mathbb{Z}_2^n) \overset{\phi_{n+1}}{\cong} \mathrm{Im}\ \ \phi_{n+1} \hookrightarrow \bar{\mathcal{F}}_{n+1} \overset{D}{\cong} \mathfrak{B}_{n-2} \xrightarrow{\partial_{n-2}} \mathfrak{B}_{n-3} \tag{1}$$

is exact. Hence,

$$\mathcal{Z}_{n+1}(\mathbb{Z}_2^n) \overset{\phi_{n+1}}{\cong} \mathrm{Im}\ \phi_{n+1} \overset{D}{\cong} \ker\partial_{n-2} = H_{n-2}(\mathfrak{B};\mathbb{Z}_2).$$

Faithful $\mathbb{Z}_2^n$-rep.'s

Definition (faithful representation)

Let $\tau = \rho_1 \cdots \rho_m$ be an m-dim $\mathbb{Z}_2^n$-rep., where $\rho_i \in \mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$ is the irreducible sub-rep of τ, and the product corresponds to direct sum of rep.'s.
τ is called *faithful*, if all ρ_i's are nontrivial and they span $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$.

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Faithful tangent representations

Let M be an m-dimensional smooth closed manifold admitting an effective $\mathbb{Z}_2^n$-action with isolated fixed points, representing an equivariant bordism class in $\mathcal{Z}_m(\mathbb{Z}_2^n)$. By the effectiveness of action, the tangent representation $\tau_x M$ at each fixed point x is faithful.

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Denote by

$$\mathcal{F}_m \subseteq \mathcal{R}_m(\mathbb{Z}_2^n), m \geq n$$

be the set of all m-dim faithful $\mathbb{Z}_2^n$-rep.'s. Let $\bar{\mathcal{F}}_m$ be the subspace spanned by $\mathcal{F}_m$. Of course, $\mathrm{Im}\ \phi_m \subseteq \bar{\mathcal{F}}_m$.

Dualization in the papers on $\mathcal{Z}_n(\mathbb{Z}_2^n)$

- Let $\tau = \rho_1 \cdots \rho_n \in \mathcal{F}_n$. Then $\{\rho_1, \cdots, \rho_n\}$ forms a basis of $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$.
- Let $\sigma_\tau = \{\alpha_1, \cdots, \alpha_n\} \subseteq \mathbb{Z}_2^n$ be its dual.
- Define dualization $D : \mathcal{F}_n \longrightarrow C_{n-1}(X(\mathbb{Z}_2^n))$ via $D(\tau) = \sigma_\tau$.

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Theorem (Lü-Tan(2014))

Let $\mathcal{A} \subseteq \mathcal{F}_n$ be nonempty. Then

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Corollary (Chen-Lü-Tan(2025))

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is exact, where $X(\mathbb{Z}_2^n)$ is the universal complex whose simplexes are independent subsets of $\mathbb{Z}_2^n$. Therefore, $\mathcal{Z}_n(\mathbb{Z}_2^n) \cong \tilde{H}_{n-1}(X(\mathbb{Z}_2^n), \mathbb{Z}_2)$ with dimension $A_n = (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1-i} \frac{1}{(i+1)!} \prod_{j=0}^{i} (2^n - 2^j)$.

Equivalence relation $\sim_\rho$

Definition

- Let ρ be an irreducible non-trivial sub-rep of τ. rank $\ker \rho = n-1$. Denote by

$$\tau\big|_{\ker\rho}$$

  the quotient of the restriction $\mathrm{Res}^{\mathbb{Z}_2^n}_{\ker\rho}\tau$ by a trivial 1-dimesnional sub-rep.
- $\dim \tau\big|_{\ker\rho} - \text{rank } \ker\rho = (m-1)-(n-1) = m-n$, same as τ.

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Equivalently, $\tau = \rho\rho_2\cdots\rho_m \sim_\rho \tau' = \rho\rho_2'\cdots\rho_m'$ if $\exists \pi \in S_{m-1}$ such that $\rho_i' = \rho_{\pi(i)}$ or $\rho_i' = \rho_{\pi(i)} + \rho$ in $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$.

Dualization of a faithful representation $\tau \in \mathcal{F}_{n+1}$

- Unlike the case $m = n$, the factors of $\tau = \rho_0\rho_1 \cdots \rho_n$ no longer form a unique basis for $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$.
- A construction of $D(\tau)$ based on any single basis would be both non-canonical and inadequate for a well-defined duality framework.
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- Instead, the proper definition must incorporate the collective data from all possible bases formed by the factors of τ.
- Let $\tau = \rho_0\rho_1 \cdots \rho_n \in \mathcal{F}_{n+1}$, where $\mathbf{b} = \{\rho_1, \cdots, \rho_n\}$ forms a basis of $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$ and $\rho_0 = \rho_1 + \cdots + \rho_{p+1}$, for some p.
- $\mathbf{b} \overset{\text{dual}}{\longleftrightarrow} \sigma^p \cup \sigma^{n-p-2} = \{\alpha_1, \cdots, \alpha_{p+1}\} \cup \{\alpha_{p+2}, \cdots, \alpha_n\}$.
- Bases arising from τ: $\mathbf{b} \setminus \{\rho_i\} \cup \{\rho_0\} \overset{\text{dual}}{\longleftrightarrow} \sigma_i^p \cup \sigma^{n-p-2}$, where $\sigma_i^p = (\sigma^p \setminus \{\alpha_i\} + \alpha_i) \cup \{\alpha_i\} = \{\alpha_{1i}, \cdots, \alpha_{i-1,i}, \alpha_i, \alpha_{i+1,i}, \cdots, \alpha_{p+1,i}\}$, $i \in [p+1]$.

Dualization of a $\mathbb{Z}_2^n$-rep with dim $n+1$

Definition (Dual of τ)

With the notation above, define

$$D(\tau) \overset{\triangle}{=} [\sigma^p] \otimes \sigma^{n-p-2} = (\sigma^p + \sum_{i=1}^{p+1} \sigma_i^p) \otimes \sigma^{n-p-2}.$$

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- $\tau \sim_\rho \tau' \iff \tau|_{\ker \rho} = \tau'|_{\ker \rho} \iff D(\tau|_{\ker \rho}) = D(\tau'|_{\ker \rho})$
- $D(\tau) \in D_p \otimes C_{n-p-2}$, where
    - C_{n-p-2} is the $(n-p-2)$-th group of augmented simplicial chain complex of $X(\mathbb{Z}_2^n)$,
    - $D_p \subseteq C_p$ is generated by $[\sigma^p]$'s.
- $D(\tau|_{\ker \rho}) = [\sigma^p \setminus \{\alpha\}] \otimes (\sigma^{n-p-2} \setminus \{\alpha\})$, where $\sigma^p \cup \sigma^{n-p-2}$ is the dual of a basis containing ρ and arising from τ, α is the vector in the dual corresponding to ρ.

Summary for dualization of a representation

Let τ be a $\mathbb{Z}_2^n$-representation with dimension $n+1$. Suppose all the irreducible sub-rep.'s of τ Span Hom$(\mathbb{Z}_2^n, \mathbb{Z}_2)$.

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where $\sigma \in \mathrm{Lk}\, \omega$ and $\sigma \cup \omega$ is the dual of a basis arising from τ.

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A key observation

- the terms in $[\sigma]$ have form $(\sigma \setminus \{\alpha\} + \alpha) \cup \{\alpha\}$, where $\alpha \in \sigma \in X(\mathbb{Z}_2^n)$.
- $\mathrm{Span}\big((\sigma \setminus \{\alpha\} + \alpha) \cup \{\alpha\}\big) = \mathrm{Span}\ \sigma \subseteq \mathbb{Z}_2^n$.
- Hence $\mathrm{Lk}\big((\sigma \setminus \{\alpha\} + \alpha) \cup \{\alpha\}\big) = \mathrm{Lk}\ \sigma \triangleq \mathrm{Lk}[\sigma]$.

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Let τ be a $\mathbb{Z}_2^n$-representation with dimension $n+1$. Suppose all the irreducible sub-rep.'s of τ Span $\mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$.

$$D(\tau) = [\sigma] \otimes \omega = \begin{cases} \emptyset \otimes \omega, & \tau \text{ admits a trivial sub-rep,} \\ [\sigma] \otimes \omega, & \text{otherwise} \end{cases}$$

where $\sigma \in \mathrm{Lk}\ \omega$ and $\sigma \cup \omega$ is the dual of a basis arising from τ.

A key observation

- the terms in $[\sigma]$ have form $(\sigma \setminus \{\alpha\} + \alpha) \cup \{\alpha\}$, where $\alpha \in \sigma \in X(\mathbb{Z}_2^n)$.
- $\mathrm{Span}\big((\sigma \setminus \{\alpha\} + \alpha) \cup \{\alpha\}\big) = \mathrm{Span}\ \sigma \subseteq \mathbb{Z}_2^n$.
- Hence $\mathrm{Lk}\big((\sigma \setminus \{\alpha\} + \alpha) \cup \{\alpha\}\big) = \mathrm{Lk}\ \sigma \triangleq \mathrm{Lk}[\sigma]$.
- $\omega \in \mathrm{Lk}[\sigma]$.

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- Hence $\mathrm{Lk}\big((\sigma \setminus \{\alpha\} + \alpha) \cup \{\alpha\}\big) = \mathrm{Lk}\ \sigma \triangleq \mathrm{Lk}[\sigma]$.
- $\omega \in \mathrm{Lk}[\sigma]$.
- $\mathrm{Lk}[\sigma] \simeq (\bigvee S^k)^{A_{p,n}}$, where $k = n - |\sigma| - 1$.

LLS detection method

Theorem (LLS detection method)

Let $\mathcal{A} \subseteq \mathcal{F}_{n+1}$ be nonempty. Then the following statements are equivalent.

(1) $\sum_{\tau\in\mathcal{A}} \tau \in$ *Im* ϕ_{n+1}.

(2) *For any nontrivial $\rho \in \mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$ such that $\mathcal{A}_\rho \neq \emptyset$, $\mathcal{A}_\rho \subseteq \mathcal{A}$ satisfies that for each equivalence class $\mathcal{A}_{\rho,i}$ in the quotient set $\mathcal{A}_\rho/\sim_\rho$,*

- $|\mathcal{A}_{\rho,i}| \equiv 0 \pmod 2$, *and*
- *if $\chi_\rho(\mathcal{A}_{\rho,i}) = 2$, then for arbitary nontrivial element $\beta \in \mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{Z}_2)$,*

$$\sum_{\tau\in\mathcal{A}_{\rho,i}} \chi_\beta(\tau) \equiv 0 \pmod 2.$$

Dualization of LLS

Theorem

Let $\mathcal{A} \subseteq \mathcal{F}_{n+1}$ be nonempty. Then

$$\sum_{\tau \in \mathcal{A}} \tau \in \mathit{Im}\ \phi_{n+1} \Longleftrightarrow \sum_{\tau \in \mathcal{A}} \partial D(\tau) = 0,$$

where

$$\partial D(\tau) = \begin{cases} \sum_{\rho \in (\tau)} D(\tau|_{\ker \rho}), & \tau \text{ is square-free as a monomial}, \\ \alpha_1 D(\tau|_{\ker \rho_1}) + \sum_{i=2}^{n} D(\tau|_{\ker \rho_i}), & \tau = \rho_1^2 \rho_2 \cdots \rho_n. \end{cases}$$

where α_1 is the vector in the dual basis of $\{\rho_1, \cdots, \rho_n\}$ corresponding to ρ_1.

Key of the proof of dualizaiton theorem

$$\begin{bmatrix} 1 & \epsilon_2 & \epsilon_3 & \cdots & \epsilon_n \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}^{-1,T} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \epsilon_2 & 1 & 0 & \cdots & 0 \\ \epsilon_3 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ \epsilon_n & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\tau = \rho_1^2\rho_2 \cdots \rho_n \sim_{\rho_1} \tau' = \rho_1^2\rho_2' \cdots \rho_n'$$

where $\rho_i' = \rho_i + \epsilon_i\rho_1, i = 2, \cdots, n$.
The dual of bases are

$$\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$$

and

$$\{\alpha_1', \alpha_2, \cdots, \alpha_n\}$$

where $\alpha_1' = \alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon_n\alpha_n$.

Computation of the dimensions

Differentials in $\mathfrak{D}$

- $\mathfrak{D}_p \triangleq \mathsf{Span}_{\mathbb{Z}_2}\{[\sigma^p] \mid \sigma^p \in X(\mathbb{Z}_2^n), |\sigma^p| = p+1\}, p = 0, 1, \cdots, n-1;$
- $\mathfrak{D}_{-1} \triangleq \mathbb{Z}_2^n\langle\emptyset\rangle;$
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- $\mathfrak{D}_{-1} \triangleq \mathbb{Z}_2^n\langle\emptyset\rangle;$
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- For $p \in [n-1]$, set $d_p^D : \mathfrak{D}_p \to \mathfrak{D}_{p-1}$ by

$$d_p^D([\sigma^p]) = [\sigma_1^p \setminus \{\alpha_1\}] + \sum_{i=1}^{p+1}[\sigma^p \setminus \{\alpha_i\}].$$

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- For $p = 0$, define $d_0^D : D_0 \to D_{-1}$ via $d_0^D([\{\alpha\}]) = \alpha$ for each $\alpha \in X_0$.

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- For $p = 0$, define $d_0^D : D_0 \to D_{-1}$ via $d_0^D([\{\alpha\}]) = \alpha$ for each $\alpha \in X_0$.
- $d_{p-1}^D \circ d_p^D = 0$. Hence $(\mathfrak{D}, d^D)$ is a chain complex.

Construction of $\mathfrak{B}$

- $\mathfrak{D} \otimes \mathfrak{C}$, the tensor product of two chain complex is also a chain complex.
- Set $B_{p,q} = \mathsf{Span}\{[\sigma^p] \otimes \sigma^q \mid \sigma^q \in \mathsf{Lk}\sigma^p\} \subseteq \mathfrak{D}_p \otimes \mathfrak{C}_q$.

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- $\mathfrak{B}$ is defined to be the total complex of the double complex $\{B_{*,*}\}$.
- The top chain group $\mathfrak{B}_{n-2}$ is isomorphic(via the dualization D) to the linear space $\bar{\mathcal{F}}_{n+1}$ generated by faithful representations.
- the sequence

$$0 \longrightarrow \mathcal{Z}_{n+1}(\mathbb{Z}_2^n) \overset{\phi_{n+1}}{\cong} \mathrm{Im}\ \ \phi_{n+1} \hookrightarrow \bar{\mathcal{F}}_{n+1} \overset{D}{\cong} \mathfrak{B}_{n-2} \xrightarrow{\partial_{n-2}} \mathfrak{B}_{n-3}$$

is exact.

The double complex $(B_{*,*}, d_*^D \otimes 1, 1 \otimes d_*^C)$

Page E^0:

$\cdots$				$\cdots$		
$n-1$	0	0	0	$\cdots$	0	0
$n-2$	$\mathbb{Z}_2^n \otimes C_{n-2}$	$B_{0,n-2}$	0	$\cdots$	0	0
$n-3$	$\mathbb{Z}_2^n \otimes C_{n-3}$	$B_{0,n-3}$	$B_{1,n-3}$	$\cdots$	0	0
$\cdots$				$\cdots$		
0	$\mathbb{Z}_2^n \otimes C_0$	$B_{0,0}$	$B_{1,0}$	$\cdots$	$B_{n-2,0}$	0
-1	$\mathbb{Z}_2^n \otimes \mathbb{Z}_2$	$D_0 \otimes \mathbb{Z}_2$	$D_1 \otimes \mathbb{Z}_2$	$\cdots$	$D_{n-2} \otimes \mathbb{Z}_2$	$D_{n-1} \otimes \mathbb{Z}_2$
(p,q)	-1	0	1	$\cdots$	$n-2$	$n-1$

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Page E^0:

$\cdots$				$\cdots$		
$n-1$	0	0	0	$\cdots$	0	0
$n-2$	$\mathbb{Z}_2^n \otimes C_{n-2}$	$B_{0,n-2}$	0	$\cdots$	0	0
$n-3$	$\mathbb{Z}_2^n \otimes C_{n-3}$	$B_{0,n-3}$	$B_{1,n-3}$	$\cdots$	0	0
$\cdots$				$\cdots$		
0	$\mathbb{Z}_2^n \otimes C_0$	$B_{0,0}$	$B_{1,0}$	$\cdots$	$B_{n-2,0}$	0
-1	$\mathbb{Z}_2^n \otimes \mathbb{Z}_2$	$D_0 \otimes \mathbb{Z}_2$	$D_1 \otimes \mathbb{Z}_2$	$\cdots$	$D_{n-2} \otimes \mathbb{Z}_2$	$D_{n-1} \otimes \mathbb{Z}_2$
(p,q)	-1	0	1	$\cdots$	$n-2$	$n-1$

- $1 \otimes d_q^C : B_{p,q} \to B_{p,q-1}$.

The double complex $(B_{*,*}, d^D_* \otimes 1, 1 \otimes d^C_*)$

Page E^0:

$\cdots$				$\cdots$		
$n-1$	0	0	0	$\cdots$	0	0
$n-2$	$\mathbb{Z}_2^n \otimes C_{n-2}$	$B_{0,n-2}$	0	$\cdots$	0	0
$n-3$	$\mathbb{Z}_2^n \otimes C_{n-3}$	$B_{0,n-3}$	$B_{1,n-3}$	$\cdots$	0	0
$\cdots$				$\cdots$		
0	$\mathbb{Z}_2^n \otimes C_0$	$B_{0,0}$	$B_{1,0}$	$\cdots$	$B_{n-2,0}$	0
-1	$\mathbb{Z}_2^n \otimes \mathbb{Z}_2$	$D_0 \otimes \mathbb{Z}_2$	$D_1 \otimes \mathbb{Z}_2$	$\cdots$	$D_{n-2} \otimes \mathbb{Z}_2$	$D_{n-1} \otimes \mathbb{Z}_2$
(p,q)	-1	0	1	$\cdots$	$n-2$	$n-1$

- $1 \otimes d^C_q : B_{p,q} \to B_{p,q-1}$.
- $B_{p,q} = \mathsf{Span}\{[\sigma^p] \otimes \omega^q \mid \omega^q \in \mathsf{Lk}\sigma^p\} = \bigoplus_{[\sigma^p]} C_q(\mathsf{Lk}\ \sigma^p)$.

The double complex $(B_{*,*}, d_*^D \otimes 1, 1 \otimes d_*^C)$

Page E^0:

$\cdots$				$\cdots$		
$n-1$	0	0	0	$\cdots$	0	0
$n-2$	$\mathbb{Z}_2^n \otimes C_{n-2}$	$B_{0,n-2}$	0	$\cdots$	0	0
$n-3$	$\mathbb{Z}_2^n \otimes C_{n-3}$	$B_{0,n-3}$	$B_{1,n-3}$	$\cdots$	0	0
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(p,q)	-1	0	1	$\cdots$	$n-2$	$n-1$

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- $\mathsf{Lk}\ \sigma^p \simeq (\bigvee S^{n-p-2})^{A_{p,n}}$.

First page of the spectral sequence

E^1 **Page**:

$\cdots$				$\cdots$		
$n-1$	0	0	0	$\cdots$	0	0
$n-2$	$E^1_{-1,n-2}$	$E^1_{0,n-2}$	0	$\cdots$	0	0
$n-3$	0	0	$E^1_{1,n-3}$	$\cdots$	0	0
$\cdots$				$\cdots$		
0	0	0	0	$\cdots$	$E^1_{n-2,0}$	0
-1	0	0	0	$\cdots$	0	$D_{n-1}\otimes\mathbb{Z}_2$
(p,q)	-1	0	1	$\cdots$	$n-2$	$n-1$

- $E_{-1,n-2}=\mathbb{Z}_2^n\otimes\ker d^C_{n-2}=\mathbb{Z}_2^n\otimes\mathsf{Im}\ d^C_{n-1}$;
- $E_{p,n-2-p}=\bigoplus_{[\sigma^p]}H_{n-2-p}(\mathsf{Lk}\ \sigma^p;\mathbb{Z}_2), p=0,1,\cdots,n-2.$
- There is only one nontrivial differential $d_1=d^D_0\otimes 1: E^1_{0,n-2}\to E^1_{-1,n-2}$.

Second page of the spectral sequence

E^2 **Page**:

$\cdots$				$\cdots$		
$n-1$	0	0	0	$\cdots$	0	0
$n-2$	coker d_1	$\ker d_1$	0	$\cdots$	0	0
$n-3$	0	0	$E^1_{1,n-3}$	$\cdots$	0	0
$\cdots$				$\cdots$		
0	0	0	0	$\cdots$	$E^1_{n-2,0}$	0
-1	0	0	0	$\cdots$	0	$D_{n-1} \otimes \mathbb{Z}_2$
(p,q)	-1	0	1	$\cdots$	$n-2$	$n-1$

Second page of the spectral sequence

E^2 **Page**:

$\cdots$				$\cdots$		
$n-1$	0	0	0	$\cdots$	0	0
$n-2$	coker d_1	$\ker d_1$	0	$\cdots$	0	0
$n-3$	0	0	$E^1_{1,n-3}$	$\cdots$	0	0
$\cdots$				$\cdots$		
0	0	0	0	$\cdots$	$E^1_{n-2,0}$	0
-1	0	0	0	$\cdots$	0	$D_{n-1} \otimes \mathbb{Z}_2$
(p,q)	-1	0	1	$\cdots$	$n-2$	$n-1$

proposition

The only one nontrivial differential $d_2 : E^1_{1,n-3} \to$ coker d_1 is surjective.

Dimension computation: Spectral sequence

E^3 **Page**:

$\cdots$					$\cdots$		
$n-1$	0	0	0	0	$\cdots$	0	0
$n-2$	0	$\ker d_1$	0	0	$\cdots$	0	0
$n-3$	0	0	$\ker d_2$	0	$\cdots$	0	0
$n-4$	0	0	0	$E^1_{2,n-4}$	$\cdots$	0	0
$\cdots$					$\cdots$		
0	0	0	0	0	$\cdots$	$E^1_{n-2,0}$	0
-1	0	0	0	0	$\cdots$	0	$D_{n-1} \otimes \mathbb{Z}_2$
(p,q)	-1	0	1	2	$\cdots$	$n-2$	$n-1$

Dimension formula

Theorem

$$\dim \mathcal{Z}_{n+1}(\mathbb{Z}_2^n) = A_{0,n} \cdot f_0 + \sum_{p=1}^{n-2} \frac{A_{p,n} \cdot f_p}{p+2} - (n - \frac{1}{n+1}) f_{n-1} + n \cdot A_n$$

where $A_1 = 0, A_{0,1} = 0,\ f_p = \frac{\prod_{k=0}^{p}(2^n - 2^k)}{(p+1)!}$ *for* $p \geq 0,\ A_n = (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1-i} f_i$ *for* $n > 1$, *and*

$$A_{p,n} = (-1)^{n-p-1} + \sum_{i=0}^{n-p-2} (-1)^{n-p-i} \frac{\prod_{j=0}^{i}(2^n - 2^{p+j+1})}{(i+1)!}, n > 1, 0 \leq p \leq n-2.$$

Further discussion

Further discussion

The trick on $\mathcal{Z}_m(\mathbb{Z}_2^n)$

- For $\mathcal{Z}_{n+2}(\mathbb{Z}_2^n)$: We have recently dualized the LLS for this case. The description is significantly much more complex than that for $\mathcal{Z}_{n+1}(\mathbb{Z}_2^n)$.
- It appears difficult to extend this trick to the general case.
- The primary reason is the rapidly growing complexity of the linear relationships as $m-n$ increases.

(P3)

Seek specific manifolds for equivariant bordism classes in $\mathcal{Z}_{n+1}(\mathbb{Z}_2^n)$.

Thanks for your attention!

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